Weak Field Phase Diagram for an Integer Quantum Hall Liquid

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We investigate the transition of a two-dimensional electron gas from the regime of the quantum Hall effect to the regime of weak magnetic fields for a tight-binding model. Unlike previous work, we find the following: (1) the linear field dependence of the extended-state energies is not affected by disorder, although the total density of states below each level of extended states increases with disorder strength; (2) for each Landau band and disorder strength there exists a critical field B_c below which the extended level disappears, with B_c smaller for lower Landau bands. We show how the experimental findings of level flotation and direct transition from high Landau level states to the Anderson insulating phase may be explained in light of our results.

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It is an unsolved problem regarding how a two-dimensional electron gas transforms from the quantum Hall regime [1] to the weak magnetic field regime. According to the scaling theory of localization [2] all electrons in a two-dimensional system are localized in the absence of magnetic field. In the presence of a strong magnetic field, the energy spectrum becomes a series of impurity broadened Landau bands; extended states reside in the center of each band while the states at other energies are localized. The quantum Hall effect is observed in the strong field regime, where the Hall conductance is quantized and jumps from one quantized value to another when the Fermi energy crosses an extended-state level. It is important to know the fate of the extended Landau levels as $B \to 0$. According to the conventional picture of Khmelnitskii [3] and Laughlin [4], the extended levels stay with the centers of the Landau bands at strong magnetic field, but float up in energy at small magnetic field and go to infinity as $B \to 0$. This is also a central ingredient of the theory of Kivelson, Lee, and Zhang [5] for the global phase diagram of the quantum Hall effect.

In this Letter, we propose an alternative scenario for the behavior of the extended Landau levels, which is based on our numerical results for a tight-binding model of two-dimensional electrons in a magnetic field and random potentials. In our picture, each extended level is destroyed by strong disorder at a critical magnetic field instead of floating up in energy. Using a direct calculation of the localization length for finite-size samples using a transfer matrix technique, we systematically investigate the field and disorder dependence of the energies of extended states in the regime of strong coupling between the Landau bands. We find the following: (1) the linear field dependence of the extended-state energies is not affected by disorder, although the total density of states below each extended level increases with disorder; (2) for a given Landau band there exists a critical magnetic field B_c below which the extended level disappears, with B_c smaller for lower Landau bands and for weaker disorder.

Our results can be summarized as the phase diagram presented in Fig. 1, where only the $E < 0$ part is shown because of symmetry. At strong magnetic fields, extended levels appear in the center of Landau bands [$E_n = (n + 1/2)\hbar\omega_c$, where $\omega_c = eB/mc$]. The linear field dependence of these energies is maintained as the field becomes weak, until they hit a phase boundary represented by the thick curve, where they become terminated. The states on the phase boundary are themselves extended, but carry negative Chern numbers. The phase boundary has such a shape that the higher Landau levels disappear at larger magnetic fields. As disorder strength increases, it moves towards the direction of lower energy and higher magnetic

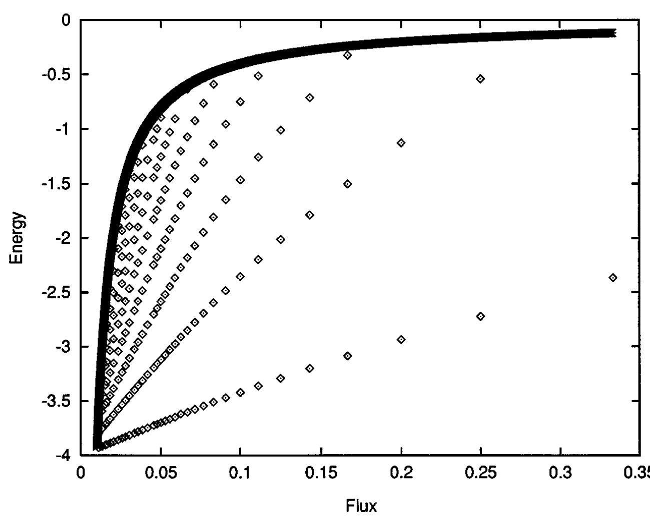

FIG. 1. Our proposed weak field phase diagram for an integer quantum Hall liquid in a lattice model. The diamonds represent the extended levels in the center of Landau bands. The thick solid line represents the boundary between quantum Hall liquid and Anderson insulator.

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field. Experimental consequences of our phase diagram will be discussed.

We first briefly outline our model and technique for calculating the localization length (a similar model was discussed in previous works [6,7]). Our two-dimensional system is a standard tight-binding square lattice with nearest neighbor hopping, in a very long strip geometry with a finite width (M). The periodic boundary condition in the width direction is used to remove the edge extended states. The disorder is modeled by the on-site white-noise potential V_{im} (i denotes the column index, m denotes the chain index) distributed uniformly from $-W/2$ to $W/2$. The magnetic field is represented in the phase of the hopping terms. The strength of the magnetic field is characterized by the flux per plaquette (ϕ) in units of magnetic flux quanta (hc/e). Unlike the closed torus geometry, our model can allow arbitrarily small values of the magnetic field. The amplitude of hopping is chosen as the unit of energy. For a specific energy E, a transfer matrix $T_i(E)$ can be set up to map the wave-function amplitudes at column $i - 1$ and i to those at column $i + 1$. Using a standard iteration algorithm [8], we can calculate the Lyapunov exponents for the transfer matrix $T_i(E)$. The localization length $\lambda_M(E)$ for energy E at finite width M is then given by the inverse of the smallest Lyapunov exponent. In our numerical calculation, we choose the sample length to be over 10^4 to achieve self-averaging.

In Fig. 2, we present the localization length at various values of the magnetic field for a finite-size sample ($M = 32$). Because of the symmetry of the lattice model, only the results for the lower energy branch are shown here. In the absence of disorder, the energy band of the tight-binding lattice ranges from $E = -4$ to $E = 4$, which breaks up into q subbands in a magnetic field with $\phi = 1/q$ (where q is an integer). The subbands in the lower energy branch correspond to Landau levels in a continuum model. Part (b) of Fig. 2 is for very weak magnetic fields, where only the lowest two Landau bands are shown for clarity. In the continuum model, each Landau level evolves into a band when disorder is turned on, with extended states residing at the center of each Landau band. The maxima of the finite-size localization length in Fig. 2 are therefore regarded as the locations of the extended states [8,9]. It is striking to see that, with disorder strength as large as $W = 1$, their energies remain linear in $B \propto \phi$, and that they intersect the parent-band minimum $E = -4.0$ as $B \to 0$, resembling the behavior of Landau levels in the continuum model in the absence of disorder.

In order to assess the finite-size effect, we carried out the same calculations for system size $M = 64$ as shown by the open circles in Fig. 2(a). One can see that, in the localized regime, the localization length is independent of sample size, while for energies close to the center of each Landau band, the localization length scales with the sample size. However, the peaks of the localization length do not shift in energy with increasing sample size. To further illustrate this point, we have carried out finite-size scaling calculations [10] for $\phi = 1/33$ with system sizes $M = 16$, 24, 32, 48, 64, and 84. The thermodynamic localization length ξ (reduced by a factor of 5 to fit into the range of the figure) for the lowest Landau band is shown in Fig. 2(b). Clearly, ξ diverges at the same energy where finite-size localization length λ_M reaches its peak. Therefore, we conclude that the finite-size effect is not important to the position of extended states which we are addressing in this paper.

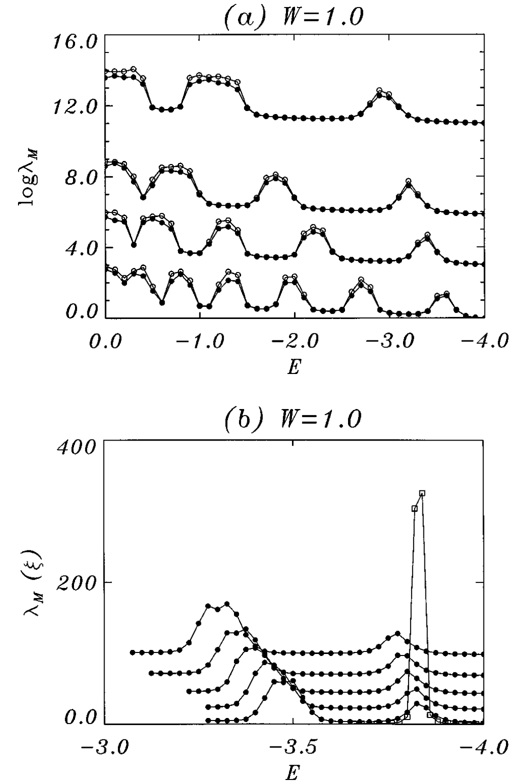

FIG. 2. Finite-size localization length (λ_M) at different weak magnetic field for fixed disorder strength $W = 1$, with lower curves corresponding to weaker field. (a) $\phi = 1/13$, $1/9$, $1/7$, $1/5$, open circles are the data for $M = 64$, filled circles for $M = 32$; (b) $\phi = 1/33$, $1/31$, $1/29$, $1/27$, $1/25$. Each curve is shifted vertically relative to the lowest one of each panel by an amount proportional to the magnetic field, in order to show the linear field dependence of the peak positions. The open squares in (b) are the thermodynamic localization length ξ (reduced by a factor of 5) for $\phi = 1/33$ obtained from finite-size scaling analysis.

We now address the effect of disorder strength on the extended states at a fixed weak magnetic field. As presented in Fig. 3, the localization length decreases as the strength of disorder increases. The peaks associated with the higher energy Landau bands are destroyed in the presence of strong enough disorder, while those of the lower Landau bands become broader. The peak for the lowest Landau band is the most robust, but it is nevertheless destroyed at stronger disorder. In the entire range of disorder strength shown in Fig. 3, the peak center positions of the Landau bands stay constant (even slightly going down) in energy without any trace of floating up, even though

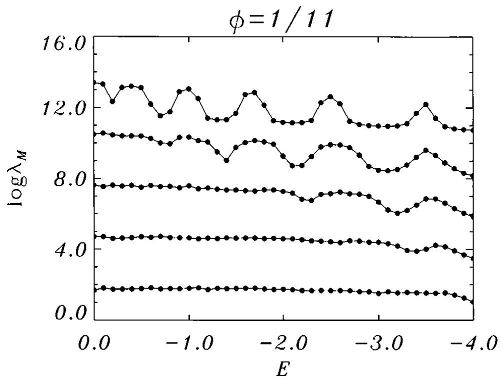

FIG. 3. Finite-size localization length (λ_M) for different disorder strength at magnetic field $\phi = 1/11$ and $W = 1, 2, 3, 4, 5$ (from the highest curve to the lowest one). The localization length for weaker disorder strength is scaled by a factor of 500 from the one for the next stronger disorder strength.

the localization lengths are changed by several orders of magnitude. These results provide additional support to the conclusion that the linear field dependence of the extended-state energies are not affected by disorder.

We believe that the localization transition here is caused by Landau level coupling effect which is more severe at weak fields. For our tight-binding model, the zero-field level broadening is [11]

$$\Gamma = \frac{W^2}{6\pi E} K\left(\frac{4t}{E}\right), \tag{1}$$

where t is the hopping amplitude which is set to unity and $K(x)$ is the complete elliptical integral of the first kind. Indeed, the peaks in Fig. 3 start to disappear when $\Gamma \simeq \omega_c$, i.e., when the Landau levels start to couple together.

There is then the question of whether and how our results conform with the conservation of Chern numbers [12]. Consider the case of flux $\phi = 1/11$, for instance; each Landau band carries a Chern number $+1$ except the band at the center ($E = 0$) which carries Chern number -10 [13,14]. There are three possibilities for the sidebands (with $+1$ Chern number) to change their Chern numbers by annihilating with the center band: (i) all the sidebands move towards center (floating up); (ii) center band splits into two -5 branches and each branch moves towards one band edge; and (iii) sidebands move towards center and center branches move towards the sides (weakly floating up). Our numerical studies confirm the second scenario, i.e., no floating up in the extended-state energies. The boundary curve in Fig. 1 represents the motion of the lower branch originally split from the center band. Each time this branch intersects with one Landau level, its Chern number reduces in magnitude by 1.

There have been a number of experimental attempts [15–19] to address the transition of delocalized states at the weak magnetic field limit. In the earlier experiments [15,16] on strongly disordered two-dimensional electron gases (2DEG), only the observation of the lowest Landau level plateau was reported ($\nu = 2$ for spin unresolved 2DEG), which is consistent with our argument that the extended states in the lowest Landau band are the most robust.

Two very recent experiments on low mobility gated GaAs/AlGaAs hetereostructure [18] and high mobility Si samples [19] have reported floating up of the delocalized states in the magnetic-field–carrier-density plane. The authors in Ref. [18] claimed that their experimental result "unambiguously" demonstrated that the "energy" of the delocalized states floats up as $B \to 0$. However, we argue that floating up of the carrier density does not necessarily imply the same behavior for the energy. Here, it is important to note that our energy is measured with respect to the band minimum of the system in the absence of the magnetic field and disorder, and no Coulomb interaction between the carriers has been taken into account. It was with respect to this definition of energy measure that Khmelnitskii [3] and Laughlin [4] discussed level flotation in their original papers.

To explain the anomalous floating up of the carrier density in the recent experiment by Glozman, Johnson, and Jiang [18], we should include the Landau level mixing effect at small magnetic field in the presence of strong disorder. The localized tail of higher Landau bands can well extend into the lower Landau bands. Therefore, the electrons have to fill up the tail of higher Landau bands before reaching the extended states in the center of the lowest Landau band, and the real filling factor could be greater than 2 (for spin unresolved 2DEG) when the quasiharmonic (QH) plateau for the lowest Landau level is observed. The same arguments apply to the filling of higher Landau bands. The point where the carrier density starts to float up is the moment where inter-Landau-level mixing shows up. The experimental fact that no floating was observed in higher mobility samples [18] strongly demonstrates the essential role of the Landau level mixing in the floating up of the carrier density.

To demonstrate this point, we show in Table I the calculated filling factor to observe the QH transition in different Landau levels for fixed magnetic field $\phi = 1/11$. Similar behavior can be seen at other values of the magnetic field. However, this "floating" from linear behavior is entirely due to the density of states overlap at strong disorder which would not affect our weak field phase diagram.

TABLE I. Filling factor for observing QH transitions, i.e., to reach the extended states in the lowest three Landau levels.

W	$n = 0$	$n = 1$	$n = 2$
0.001	0.499 ± 0.006	1.501 ± 0.006	2.500 ± 0.002
0.01	0.501 ± 0.003	1.500 ± 0.008	2.501 ± 0.006
0.1	0.511 ± 0.006	1.509 ± 0.006	2.507 ± 0.009
0.5	0.554 ± 0.010	1.544 ± 0.010	2.539 ± 0.008
1.0	0.606 ± 0.014	1.589 ± 0.010	2.581 ± 0.010
2.0	0.716 ± 0.017	1.679 ± 0.013	2.668 ± 0.013
3.0	0.819 ± 0.020	1.779 ± 0.018	2.773 ± 0.015

Density flotation is a general phenomenon beyond our model. Let us first consider the high field limit such that Landau levels are well separated. In this limit, the electron density ρ_c below E_0 (energy of the extended state for the lowest Landau level) is proportional to the Landau level degeneracy. Thus, ρ_c decreases linearly with decreasing B. At the weak field limit with strong Landau level coupling, the experimental density of states in GaAs samples can be well described by the Lorentzian form [20]

$$g(E) = \frac{1}{2\pi l^2} \sum_n \frac{\Gamma_n}{(E - E_n)^2 + \Gamma_n^2} \tag{2}$$

and

$$\rho_c = \int_{-\infty}^{E_0} g(E)\, dE \propto \omega_c \sum_n \left(1 - \frac{2}{\pi} \arctan \frac{n\omega_c}{\Gamma_n}\right), \tag{3}$$

where l is the magnetic length, ω_c is the cyclotron energy, and $E_n = (n + 1/2)\omega_c$ is the energy for the nth Landau level. Γ_n is the level broadening for the nth Landau level which is found experimentally [20] to be independent of the magnetic field. As B (or ω_c) decreases with fixed Γ_n, more terms will be contained in the summation which makes ρ_c increase. This simple calculation shows that in the high field limit ρ_c goes down linearly with decreasing B, and in the weak field limit it increases. Thus, ρ_c has to *float up* at weak enough magnetic fields. In reality, the disorder potential is Coulomb long-range type and Landau levels are much more broadened than in the short-range impurity case [21], and therefore, the effect of Landau level mixing is much more important [22].

A new prediction of our phase diagram is that it allows direct transition from quantum Hall states of higher Landau bands to the Anderson insulator. This is possible because of the phase boundary carrying a negative Chern number. One might argue that this is an artifact of our lattice model, because the states of negative Chern numbers originate from the band center, and there is not such a center in a continuum model. However, our lattice model can be more realistic than the continuum model, in that two-dimensional electrons really reside in a subband of finite width on the interface of a semiconductor heterostructure. In fact, Wang *et al.* [16] indicated briefly near the end of their paper that they saw a direct transition to the Anderson insulator from the quantum Hall state with two Landau bands occupied. This transition might also be observed in another recent experiment [19].

In conclusion, we have proposed an alternative phase diagram for the integer quantum Hall system in the weak field limit. We have demonstrated that there exists a critical magnetic field below which a Landau level is destroyed. Until this critical field, the energy of extended states in the Landau band maintains the linear field dependence found in the zero disorder case. The lower Landau levels are more robust than the higher ones, in that their critical fields are lower, and they can overcome a larger disorder strength. In the strong disorder limit, Landau level mixing effects can contribute to the floating up of the carrier density even though the energies of the extended states never float up.

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