

Self-Organized Criticality and Synchronization in a Forest-Fire Model

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(Received 6 April 1995)

Depending on the rule for tree growth, the forest-fire model shows either self-organized criticality with rule-dependent exponents, synchronization, or an intermediate behavior. This is shown analytically for the one-dimensional system, and may hold also in higher dimensions.

PACS numbers: 64.60.Lx, 05.40.+j, 05.70.Jk, 05.70.Ln

During the past years, systems which exhibit self-organized criticality (SOC) have attracted much attention, since they might explain part of the abundance of fractal structures in nature [1]. Their common features are slow driving or energy input (e.g., dropping of sand grains, increase of strain, growing of trees) and rare dissipation events which are instantaneous on the time scale of driving (e.g., avalanches [1], earthquakes [2], fires [3]). In the stationary state, the size distribution of dissipation events obeys a power law, irrespective of initial conditions and without the need to fine-tune parameters.

While the critical behavior of the sandpile model is relatively well understood, it is, in general, not clear under which conditions SOC behavior occurs. Models of the above-mentioned type show a variety of different phenomena, depending on details of the model rules. When, e.g., the form of the driving force in the earthquake model [2] is modified, a transition from SOC to synchronization (activity bursts covering the whole system) is found [4]. When in the same model the boundary conditions are changed from open to periodic, the SOC behavior is replaced by a periodic state [5]. Another earthquake model, which includes inertia, shows a superposition of a power law and a peak at large avalanches, where most of the energy is released [6]. A continuous, deterministic forest-fire model [7] shows periodic behavior or finite avalanches, depending on the value of a parameter [8].

Another question in the context of SOC deals with the universality of the critical exponents. The critical behavior of the sandpile model seems to be robust with respect to various changes, and the critical exponents of the forest-fire model do not depend on the lattice symmetry [9,10] or the presence of a further parameter [11]. In contrast, the critical exponents of the earthquake model depend continuously on the degree of energy conservation [2,12]. A similar dependence of critical exponents on the model parameter is found in a new version of the forest-fire model with tree conservation [13].

Since a great part of the above-mentioned work is performed numerically, there is substantial need for analytically understanding these various phenomena. In this paper, I study analytically a generalized version of the forest-fire model in one dimension. Different rules

for tree growth lead either to SOC behavior with rule-dependent exponents, to synchronization, or to an intermediate state with a superposition of a power law and a peak at large fires. These results are confirmed by computer simulations. Similar behavior is expected in higher dimensions.

The model is defined as follows: Each site in a one-dimensional system of length L is either occupied by a tree or it is empty. Trees on neighboring sites belong to the same tree cluster. During one time step of size dt , each tree is struck by lightning with probability f . Trees struck by lightning and all other trees in the same cluster burn down and turn to empty sites immediately. Empty sites have a distribution of life times $P(\tau)$, i.e., $P(\tau)d\tau$ is the probability that a site which just burned down will remain empty exactly until the time τ . The probability that a site which has been empty for a time τ becomes occupied by a tree during the next time unit $d\tau$ is therefore $P(\tau)d\tau/\int_{\tau}^{\infty} P(t)dt$. It is reasonable to require that the mean life time $\bar{\tau}$ of empty sites is finite: $\bar{\tau} = \int_0^{\infty} P(\tau)\tau d\tau < \infty$. This means that $P(\tau)$ decays faster than τ^{-2} for large τ . The condition that burning is fast compared to tree growth places the model in the class of systems with slow driving and instantaneous dissipation events. The mean number of trees growing per unit time is $L(1 - \rho)/\bar{\tau}$, and the mean number of lightning strokes per unit time is $Lf\rho$, where ρ is the mean tree density in the stationary state. The mean number of trees destroyed per lightning stroke consequently is

$$\bar{s} = (1 - \rho)/f\bar{\tau}\rho. \quad (1)$$

When f becomes very small, the mean number of trees destroyed by a fire is large (i.e., of the order $1/f\bar{\tau}$). If there are only large fires but no small ones, large parts of the system burn down simultaneously and may therefore be synchronized. If there are fires of all sizes up to a cutoff size, the system is close to a critical point and shows scaling over many orders of magnitude. In this paper, we always assume that the system size L is so large that no finite-size effects occur. The cutoff is then a function of f and $P(\tau)$ [see Eq. (3) below].

In the original forest-fire model, each empty site becomes occupied by a tree with probability p , and

consequently $P(\tau) = p \exp(-p\tau)$. In the limit $f/p \rightarrow 0$, the size distribution of forest clusters in the stationary state is then essentially a power law with a cutoff at sizes of the order $(p/f)/\ln(p/f)$ [14]. In the following, we will determine analytically the size distribution of forest clusters for different $P(\tau)$.

To this purpose, let us consider a string of k neighboring sites in the system. If the lightning probability f is sufficiently small, lightning does not strike this string before all its trees are grown. Starting with a completely empty state, the string passes, therefore, through a cycle. Trees grow on the string until it is completely occupied by trees. Then the forest in the neighborhood of the string will also be quite dense. The forest on the string is part of a forest cluster which is much larger than k . Eventually that cluster becomes so large that it is struck by lightning with a nonvanishing probability. Then the forest cluster burns down, and the string again becomes completely empty, and the cycle restarts. The mean time $T(k)$, which it takes for k trees to grow on k empty sites, satisfies

$$\int_{T(k)}^{\infty} P(\tau) d\tau \propto 1/k. \quad (2)$$

This equation can, e.g., be obtained from the condition that the tree density is larger than $1 - 1/k$ for $T = T(k)$. A more rigorous derivation gives the same result. We assumed that the probability that lightning strikes the string during that time is negligible. The maximum string size for which our considerations are valid is therefore given by the condition

$$k_{\max} f \int_0^{T(k_{\max})} [T(k_{\max}) - \tau] P(\tau) d\tau \ll 1. \quad (3)$$

After time t , an initially empty string of $k < k_{\max}$ sites is occupied by $m(t) = k \int_0^t P(\tau) d\tau$ trees on an average. For not too small values of m , fluctuations around this mean value are relatively small. The probability $\mathcal{P}_k(m)$ to find m trees on a string of $k < k_{\max}$ sites therefore satisfies for $1 \ll m < k$

$$\mathcal{P}_k(m) \propto [(dm/dt)_{t=t_{mk}}]^{-1} = [kP(t_{mk})]^{-1}, \quad (4)$$

where t_{mk} is the time after which the initially empty string is occupied by m trees. Let $n(s)$ be the number of clusters of s trees, divided by the number of sites L . $n(s)$ is identical to the probability that a string of $s + 2$ neighboring sites is occupied by s neighboring trees, with one empty site at each end,

$$n(s) = P_{s+2}(s) / \binom{s+2}{s} \approx 2P_{s+2}(s)/s^2. \quad (5)$$

For small s , there are deviations from this law due to the discreteness of the lattice. Remember that the result Eq. (5) is valid only for $s < s_{\max} = k_{\max} - 2$. For larger s , clusters are struck by lightning with a nonvanishing probability before they can grow to larger clusters, and therefore Eq. (5) does not hold any more.

The mean number of trees destroyed per lightning stroke can be expressed in terms of the cluster size distribution,

$$\bar{s} = \sum_1^{\infty} s^2 n(s) / \rho = \sum_1^{s_{\max}} s^2 n(s) / \rho + \sum_{s_{\max}+1}^{\infty} s^2 n(s) / \rho. \quad (6)$$

Here, we have to distinguish two different situations: Depending on the precise form of $n(s)$, the second term can be neglected in the limit $f \rightarrow 0$ with respect to the first term, or it dominates. In the first case, the system has fires of any size and therefore is in a critical state, and in the second case, the dynamics are dominated by large fires. The first case will occur if $P(\tau)$ has a long-time tail. Then the neighborhood of the tree which is struck by lightning is likely to contain surviving empty sites which eventually stop a fire. The forest density approaches the value 1 in the limit $f \rightarrow 0$, since otherwise $\bar{s}$ could not diverge. The second case will occur if tree growth is finished after a finite time. Then large regions of the system will become simultaneously occupied by a dense forest and burn down together. In this case the mean forest density will be smaller than 1.

To make these statements more precise, let us specify the different possible cases.

(i) $P(\tau)$ has a long-time tail, i.e., $P(\tau) \propto \tau^{-\alpha}$ for large τ , with $\alpha > 2$. Sites which have been empty for a time t become occupied by a tree during the next time step with probability $(\alpha - 1)dt/t$. From Eqs. (2) and (3) we find $T(k) \propto k^{1/(\alpha-1)}$ and $s_{\max} \propto f^{-1+1/\alpha}$. With Eqs. (4) and (5) we obtain the size distribution of forest clusters

$$n(s) \propto s^{-2+1/(\alpha-1)}.$$

The forest density $\rho = \sum_{s=1}^{\infty} s n(s)$ cannot be larger than 1, and consequently $n(s)$ assumes a normalization factor $s_{\max}^{-1/(\alpha-1)} \propto f^{1/\alpha}$.

With Eq. (6) we find $\bar{s} \propto s_{\max}$, which leads with Eq. (1) to $1 - \rho \propto f^{1/\alpha}$. These results represent SOC behavior, and the values of the exponents depend on α , i.e., they are nonuniversal. The analytic results are confirmed by simulations. Figure 1 shows the collapsed curves for the size distribution $sn(s)$ of fires for $L = 10\,000$ and $\alpha = 3$, for different values of f , confirming the relations for $n(s)$, for $s_{\max}$, and the normalization factor $\propto f^{1/\alpha}$. Figure 2 shows the density of empty sites as a function of f , confirming the relation $1 - \rho \propto f^{1/\alpha}$.

(ii) $P(\tau)$ decays exponentially fast: $P(\tau) \approx \exp(-\tau^\beta)$ for large τ . This situation also comprises the original forest-fire model. The exponents are obtained from case (i) by taking the limit $\alpha \rightarrow \infty$. In addition, there occur logarithmic corrections. With Eqs. (2), (3), and (5), we find $T(k) \approx [\ln(k)]^{1/\beta}$, $s_{\max} [\ln(s_{\max})]^{1/\beta} \propto 1/f$, and $n(s) \propto s^{-2}/\ln(1/f)$. For $\beta = 1$, these results have already been analytically derived and confirmed by simulations in [14].

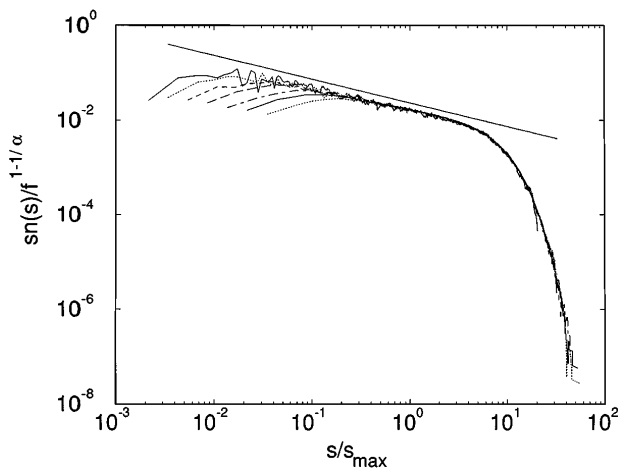


FIG. 1. Scaled size distribution $sn(s)/f^{1-1/\alpha}$ of fires as function of $s/s_{\max}$. The parameters are $L = 10000$ and $\alpha = 3$, for $1/f = 100000$ (solid line), 50000 (dotted line), 25000 (dashed line), 12500 (long dashed line), 6250 (dot-dashed line), 3125 (solid line), and 1562.5 (dotted line). The straight line has the slope -0.5 .

(iii) All trees grow within a time T_0 , i.e., $P(\tau > T_0) = 0$ and $\int_0^{T_0} P(\tau) d\tau = 1$. In this case $k_{\max}$ is essentially given by $k_{\max} \propto 1/fT_0$. The large-scale dynamics of this system can be obtained by performing a scale transformation: $k_{\max}$ sites form together a big site which becomes occupied exactly after T_0 steps, and the lightning probability for this coarse-grained system is $k_{\max}f \propto 1/T_0$. This is a model with deterministic tree growth, which shows synchronization and will be considered in the next paragraph. Before, let us take a look at the short-scale dynamics: The size distribution of clusters smaller than $s_{\max}$ is obtained in the same way as before. If $\lim_{\tau \rightarrow T_0} P(\tau) \neq 0, \infty$, we find $n(s) \propto s^{-3}$ and consequently $\sum_1^{s_{\max}} s^2 n(s)/\rho \propto \ln(1/fT_0)$. Since we know (Eq. (1)) that $\bar{s} \propto 1/fT_0$, we conclude from Eq. (6) that most fires are large and are not described

by the power law $sn(s) \propto s^{-2}$. Figure 3 shows the size distribution of fires for $P(\tau) = 1/T_0$, $T_0 = 100$, $L = 10000$, and $f = 0.00001$. In addition to the mentioned power law, there is a peak at large fires, indicating that large segments of the system burn down together.

The special case $\lim_{\tau \rightarrow 0} P(T_0 - \tau) \propto \tau^\alpha$ for $\alpha > 0$ or $-1 < \alpha < 0$ leads to $n(s) \propto s^{-3+\alpha/(1+\alpha)}$.

(iv) Last, let us consider the case of deterministic tree growth $P(\tau) = \delta(T_0 - \tau)$, which is also the coarse-grained version of situation (iii). Here it is easy to understand how synchronization arises: If two neighboring sites happen to be both occupied by a tree, they burn down during the same fire. Consequently, both trees regrow simultaneously and burn down simultaneously for all future times. So the system consists essentially of synchronized blocks which turn to trees and burn down simultaneously. Neighboring blocks join, if the block which grows first is not struck by lightning before its neighbor grows. So the number of blocks decreases with time, and their mean size increases. In the stationary state, the mean block size is so large that each block is struck by lightning immediately after it grows, so that neighboring blocks cannot join any more. This stationary state is periodic with a period T_0 . In the limit $fT_0 \rightarrow 0$, the block size becomes very large, and eventually the whole system is synchronized and fires every T_0 time steps. Figure 4 shows the number of blocks as a function of time for $L = 10000$, $f = 0.005$, and $T_0 = 50$, starting with a random state.

Generalization to higher dimensions is straightforward: It is clear that the case $P(\tau) = \delta(T_0 - \tau)$ leads to synchronization of large blocks of the system in any dimension. On the other hand, it is known that the system is SOC for $P(\tau) = p \exp(-p\tau)$. Therefore, it can be expected that when varying $P(\tau)$ the critical exponents change their values, and that finally the system becomes synchronized. However, the calculations performed in this paper cannot be applied to higher dimensions. In contrast one dimension, fire can now burn regions which

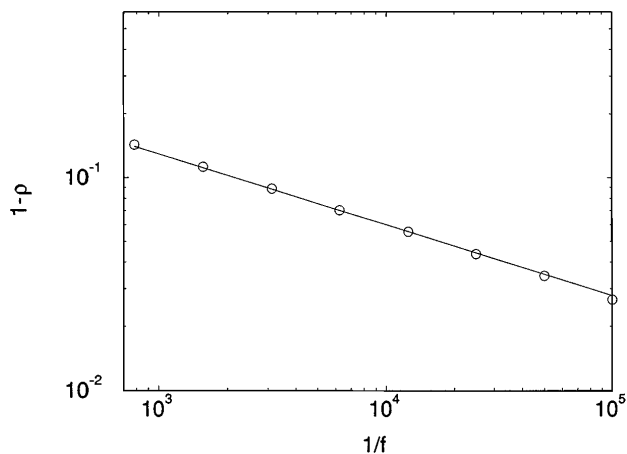


FIG. 2. Density of empty sites as a function of $1/f$ for $L = 10000$ and $\alpha = 3$. The solid line has the slope $-1/3$.

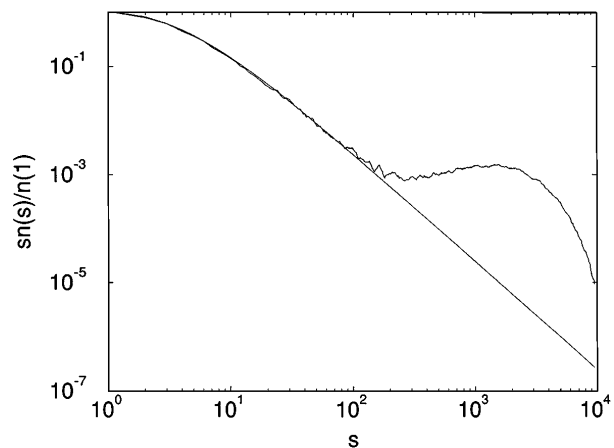


FIG. 3. Normalized size distribution of fires for $P(\tau) = 1/100$, $T_0 = 100$, $L = 10000$, and $f = 0.00001$. The smooth line is the theoretical result Eq. (5), which is valid for $s < s_{\max}$.

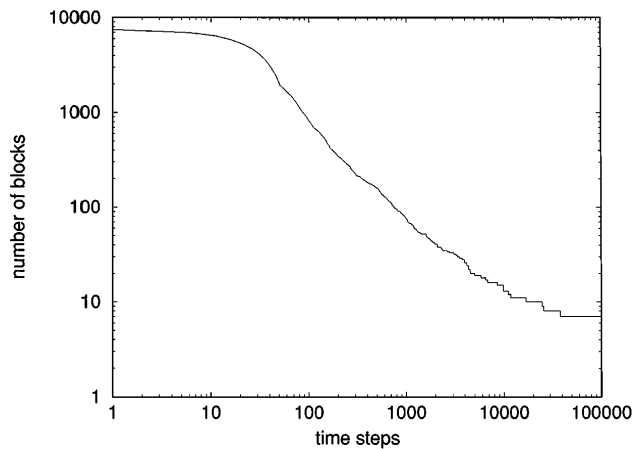


FIG. 4. Number of synchronized blocks as a function of time for $L = 10000$, $f = 0.005$, and $T_0 = 50$.

are not completely dense, and part of the trees in that region survive the fire, thus generating correlations which are not present in the one-dimensional model.

To summarize, I have shown that depending on the rule for tree growth the stationary state of the forest-fire model shows either SOC or synchronization, or a superposition of both. In the SOC state, the critical exponents vary continuously when $P(\tau)$ is changed. These results are similar to those for the earthquake model which shows SOC or synchronization depending on the form of the driving force [4]. In [5], it is shown that the SOC behavior in the earthquake model is related to the tendency of neighboring sites to synchronize. Neighboring sites which topple during the same avalanche are also likely to topple the next time during the same avalanche. The forest-fire model shows a similar feature in the SOC state: A region which has just been crossed by a fire is empty (in one dimension) or almost empty (in two dimensions). This region is, with a high probability, again covered by an almost dense forest in the moment when it is ignited. So neighboring sites which burned down during the same fire are also likely to burn down simultaneously the next time. Due to the randomness of tree growth, this “synchronization” is not perfect. The degree of synchronization increases when tree growth becomes more deterministic, first showing a peak at large fires, and finally perfect synchronization of large clusters.

When the stochastic rule for lightning strokes is replaced by a more deterministic one, the model also becomes more synchronized. Lightning might strike only trees which have a certain age or only clusters which have a certain size. In these cases, there are only large fires in the stationary state and no small ones. However, the size distribution of forest clusters smaller than a cutoff size is not considerably affected by these changes since Eq. (4) was derived under the only assumption that lightning does not strike a string of size $k < k_{\max}$ before all its trees are grown.

The results of this paper might have implications for excitable media [15] (e.g., spreading of diseases, autocatalytic chemical reactions, and propagation of electrical activity in neurons or heart muscles). These systems essentially have three states which are called quiescent, excited, and refractory. If there is an excitation, it spreads to the quiescent neighbors. After excitation, a region is refractory to further stimulation and needs some time to recover its quiescent state. The model discussed in this paper describes an excitable medium in the limit where excitation spreads fast on the scale of the refractory time, and where spontaneous excitation is rare. In cases where this limit can be realized experimentally, such a system should show either synchronization or SOC, depending on whether the transition from the refractory to the quiescent state and from the quiescent to the excited state are deterministic or stochastic.

I thank S. Clar for comments on the manuscript. This work was supported by the Deutsche Forschungsgemeinschaft (DFG) under Contract No. Dr 300/1-1, and by NSF Grant No. DMR-93-03667.

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