Synchronizing Hyperchaos with a Scalar Transmitted Signal

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Recent work has considered the possibility of exploiting the phenomenon of chaos synchronization to achieve secure communication. But theoretical and experimental models studied thus far have been limited to low dimensional systems with one positive Lyapunov exponent. We investigate chaos synchronism in high dimensional systems. In particular, in regard to applications to communication, we show that by transmitting just one scalar signal one can achieve synchronism in chaotic systems with two or more positive Lyapunov exponents.

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Synchronization of chaotic systems has in recent years become an area of active research [1–7]. Different approaches are proposed and being pursued. Among them is the technique of Pecora and Carroll [2] who show that, when a state variable from a chaotically evolving system is transmitted as an input to a replica of part of the original system, the replica subsystem (receiver) sometimes synchronizes to the original system (sender). They suggest that this phenomenon of chaos synchronism may serve as the basis for new ways to achieve secure communication. Subsequent work enriches and substantiates the claim [3–6]. In a related paper [4] it is pointed out that the receiver subsystem does not need to be a replica of part of the sender system. In fact, allowing the use of nonreplica subsystems adds flexibility and enhances synchronism.

In this work we begin by noting that theoretical as well as experimental studies reported so far concern mainly low dimensional systems with one positive Lyapunov exponent. Perez and Cerdeira [7] show that messages masked by such simple chaotic processes, once intercepted, are sometimes readily extracted. Our objective is thus the implementation of the Pecora-Carroll paradigm of chaos synchronism in high dimensional chaotic systems which takes advantage of the increased randomness and unpredictability. In such systems one generally encounters multiple positive Lyapunov exponents (a situation called hyperchaos [8]). This feature improves security by giving rise to more complex time signals, but at the same time, it also raises the question of whether by transmitting just a single scalar variable, as would be desired in a communication situation, we can still achieve synchronization. Naively, one speculates that the number of variables to be transmitted should be equal to that of positive Lyapunov exponents in order to account for the same number of unstable directions along the chaotic trajectory. In this Letter, using two examples of hyperchaos, one a differential equation and the other a discrete map, we show that such intuition is incorrect, and in fact, chaos synchronism is attained over a broad range of parameters by using a transmitted signal that is expressed as a linear combination of the original phase space variables. We proceed to argue that this approach is general and does not depend on the systems investigated.

Problem formulation.—For specificity, we assume the sender to be a chaotic system in the form

$$d\mathbf{x}(t)/dt = \mathbf{F}(\mathbf{x}(t)), \qquad (1)$$

where $\mathbf{x} \in \mathbf{R}^m$ is an m-dimensional vector. In addition, we take the signal to be transmitted as a scalar variable in the form $u(t) = \mathbf{K}^T\mathbf{x}(t) = K_1x_1(t) + K_2x_2(t) + \cdots + K_mx_m(t)$, with $\mathbf{K}$ a constant column vector. Here T denotes matrix transpose. The receiver subsystem is then written as

$$d\mathbf{y}(t)/dt = \mathbf{F}(\mathbf{y}(t)) - \mathbf{B}(v(t) - u(t)), \qquad (2)$$

where $v(t) = \mathbf{K}^T\mathbf{y}(t)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_m)^T$ is an m-dimensional constant vector.

Observe that if $\mathbf{y}(t) = \mathbf{x}(t)$ is plugged into Eq. (2), the equation is satisfied, meaning that synchronization of chaos is possible for the combined system, Eqs. (1) and (2). The question is whether this solution is attracting so that it can be realized in practice. To answer this question we evaluate the largest Lyapunov exponent for the subsystem Eq. (2) with respect to the trajectory $\mathbf{y}(t) = \mathbf{x}(t)$. Consider infinitesimal deviations of $\mathbf{y}(t)$ from $\mathbf{x}(t)$, i.e.,

$$\mathbf{y}(t) = \mathbf{x}(t) + \delta\mathbf{y}(t).$$

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From (2),

$$d\delta\mathbf{y}/dt = [\partial\mathbf{F}(\mathbf{y})/\partial\mathbf{y}|_{\mathbf{y}=\mathbf{x}} - \mathbf{B}\mathbf{K}^T]\delta\mathbf{y}. \tag{3}$$

The largest subsystem Lyapunov exponents, denoted Λ, are then given by solving (3) using a typical orbit $\mathbf{x}(t)$ for the original system (1) and a typical orientation for $\delta\mathbf{y}(0)$,

$$\Lambda = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\delta\mathbf{y}(t)|}{|\delta\mathbf{y}(0)|}. \tag{4}$$

When $\Lambda < 0$, for typical $\mathbf{y}(0) \neq \mathbf{x}(0)$,

$$\lim_{t\to\infty} |\mathbf{y}(t) - \mathbf{x}(t)| = 0. \tag{5}$$

Namely, we observe the occurrence of synchronism.

We note that it is a common practice to transmit a signal that is a phase space variable of the original system. It is found that synchronism is often not attained when this is the case. The problem becomes more acute when the original chaotic system has two or more positive Lyapunov exponents. For such systems, even the use of a nonreplica receiver subsystem as proposed in [4], which has tunable parameters in the **B** vector, can become inadequate. In this regard, our approach outlined above represents an important step in methodology toward remedying the situation. In particular, Λ in Eq. (4) is now a function of the $2m$ parameters contained in both **K** and **B** vectors. This enlarged **K**-**B** space renders greater flexibility not only in designing the characteristics of the transmission but also in choosing the correct parameter combinations to meet the condition for and enhance the performance of synchronism. Specifically, our task is now reduced to that of locating a suitable region in the **K**-**B** space for which Λ is negative. In what follows we demonstrate how this is done and illustrate why it can be done using two examples, one a four variable Rössler equation exhibiting two positive Lyapunov exponents, and the other a coupled map lattice allowing as many positive Lyapunov exponents as the system size.

Example 1.—The hyperchaotic Rössler [8] system we treat here is written as

$$dx_1/dt = -x_2 - x_3, \tag{6}$$

$$dx_2/dt = x_1 + 0.25x_2 + x_4, \tag{7}$$

$$dx_3/dt = 3.0 + x_1x_3, \tag{8}$$

$$dx_4/dt = -0.5x_3 + 0.05x_4. \tag{9}$$

Numerical evidence indicates that the attractor has two positive Lyapunov exponents, $\lambda_1 = 0.11$ and $\lambda_2 = 0.02$. Choosing $\mathbf{K} = (\sin\theta,\, 0,\, \cos\theta,\, 0)^T$ and $\mathbf{B} = 5.0(\cos\theta,\, 0,\, \sin\theta,\, 0)^T$ somewhat arbitrarily, we plot the largest receiver subsystem Lyapunov exponent Λ as a function of θ in Fig. 1. As can be seen, there is a range of θ values over which Λ is negative. In Fig. 2 we show the result of our synchronization experiment performed on the Rössler system by plotting the difference between x_1 and y_1 for $\theta = \pi/3$. Within the resolution of the figure we obtain chaos synchronism in about 60 time units. We remark that in this example, when a plain phase space variable (i.e., x_i, $i = 1, 2, 3, 4$) is sent as the input to the receiver subsystem, no synchronization is observed.

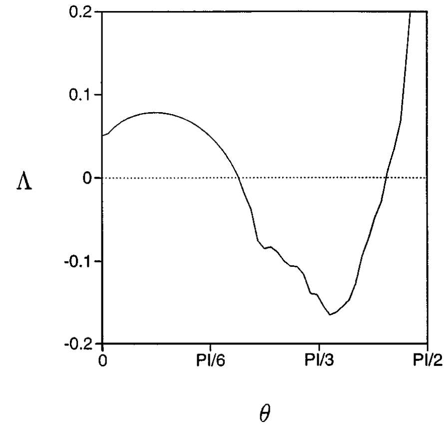

FIG. 1. The largest subsystem Lyapunov exponent Λ for the Rössler system Eqs. (6)–(9) plotted as a function of the parameter θ. It is seen that, over a substantial range of the θ value, Λ is negative, indicating synchronism between the sender and the receiver systems.

Clearly, for our approach to be successful we need to resolve the question concerning how to find regions in the **K**-**B** parameter space yielding negative Λ. We propose the following general method as a possible solution. First, for a given system, replace in Eq. (3) the vector **x** and its functions by their average values calculated on the

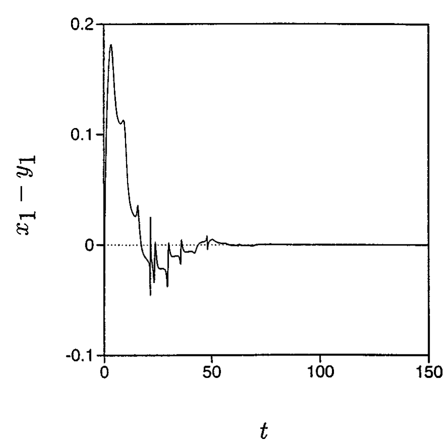

FIG. 2. Result from our numerical synchronization experiment using $\theta = \pi/3$ (see Fig. 1). The difference between the variable x_1 of the sender and the variable y_1 of the receiver asymptotes toward zero as time progresses.

original chaotic attractor. Now Eq. (3) becomes a linear equation of constant coefficients. Second, solve this linear equation and study how the eigenvalues change as the parameters B_i and K_i vary. This will give us a rough indication as to how to locate a favorable region to start our search. Third, beginning in a favorable region obtained above, integrate the full Eqs. (1) and (3), and gradually expand the parameter space exploration until finding a region where the value of Λ is negative. Our experience shows that, in general, the more positive Lyapunov exponents we have in the system, the smaller is such a region. Other ways of searching the parameter space tailored for specific systems are also attempted in our work, and they prove to be effective.

Example 2.—As stressed earlier, achieving synchronism with a scalar transmitted variable is a problem independent of the number of positive Lyapunov exponents involved. To further illustrate, we study a discrete coupled map lattice [9] that can be treated more analytically. In component form, the iteration equation analogous to Eq. (1) is

$$\begin{aligned} x_i(n+1) &= F_i(\mathbf{x}(n)) \\ &= (1 - \epsilon_{i,i-1} - \epsilon_{i,i+1}) f(x_i(n)) \\ &\quad + \epsilon_{i-1,i} f(x_{i-1}(n)) + \epsilon_{i+1,i} f(x_{i+1}(n)), \end{aligned} \tag{10}$$

where i labels the lattice site, n denotes the discrete time, $f(x)$ prescribes the local chaotic dynamics at each lattice site, and $\epsilon_{i,i+1}$ is the coupling strength from site i to site $i+1$, which is the same as that from site $i+1$ to site i, $\epsilon_{i+1,i}$, namely, $\epsilon_{i,i+1} = \epsilon_{i+1,i}$. Here we use nonuniform coupling to simulate a reaction-diffusion equation with space dependent diffusion rate. Let m be the size of the lattice and assume periodic boundary condition. The receiver equation analogous to (2) is

$$\begin{aligned} y_i(n+1) &= F_i(\mathbf{y}(n)) - B_i(v_n - u_n) = (1 - \epsilon_{i,i-1} - \epsilon_{i,i+1}) f(y_i(n)) + \epsilon_{i-1,i} f(y_{i-1}(n)) \\ &\quad + \epsilon_{i+1,i} f(y_{i+1}(n)) - B_i(v_n - u_n), \end{aligned} \tag{11}$$

where $u_n = \mathbf{K}^T\mathbf{x}(n) = K_1x_1(n) + K_2x_2(n) + \cdots + K_mx_m(n)$ is the transmitted scalar signal, and $v_n = \mathbf{K}^T\mathbf{y}(n) = K_1y_1(n) + K_2y_2(n) + \cdots + K_my_m(n)$. The linearized equation determining the subsystem Lyapunov exponents is then

$$\delta\mathbf{y}(n+1) = [\partial\mathbf{F}(\mathbf{y})/\partial\mathbf{y}|_{\mathbf{y}=\mathbf{x}(n)} - \mathbf{B}\mathbf{K}^T]\delta\mathbf{y}(n). \tag{12}$$

To facilitate our analysis we restrict ourselves to the special case of $f(x) = 2x \bmod 1$ and $m = 3$. Let $\epsilon_{1,2} = \epsilon_{2,1} = \epsilon_1$, $\epsilon_{2,3} = \epsilon_{3,2} = \epsilon_2$ and $\epsilon_{3,1} = \epsilon_{1,3} = \epsilon_3$. Under these conditions Eq. (12) becomes

$$\delta\mathbf{y}(n+1) = (\mathbf{A} - \mathbf{B}\mathbf{K}^T)\delta\mathbf{y}(n), \tag{13}$$

where $\mathbf{A}$ is a constant matrix taking the explicit form,

$$\mathbf{A} = 2\begin{pmatrix} 1-\epsilon_1-\epsilon_3 & \epsilon_1 & \epsilon_3 \\ \epsilon_1 & 1-\epsilon_1-\epsilon_2 & \epsilon_2 \\ \epsilon_3 & \epsilon_2 & 1-\epsilon_2-\epsilon_3 \end{pmatrix}.$$

For computation, we further fix $\epsilon_1 = 0.1$, $\epsilon_2 = 0.05$, and $\epsilon_3 = 0.15$. With this choice of parameters, $\mathbf{A}$'s eigenvalues are calculated to be 2.0, 1.57, and 1.23. Thus the Lyapunov exponents for the original system are $\lambda_1 = 0.69$, $\lambda_2 = 0.45$, and $\lambda_3 = 0.21$, all being positive.

Now consider the problem of achieving synchronism between Eq. (10) and Eq. (11) in the presence of three positive Lyapunov exponents by transmitting a scalar u_n. To reduce the task of searching the parameter space, we somewhat arbitrarily let $\mathbf{B} = (1, 1.5, 0)^T$ and let $\mathbf{K} = r(4, -2, 3)^T$. In Fig. 3 we plot the value of the largest subsystem Lyapunov exponent Λ as a function of r. Clearly, in the region of $3.51 < r < 4.75$, Λ becomes negative.

This example can be put on a more general footing by recalling a result from control theory [10]. Consider the following discrete control system,

$$\mathbf{z}(n+1) = \mathbf{A}\mathbf{z}(n) + \mathbf{B}p(n), \tag{14}$$

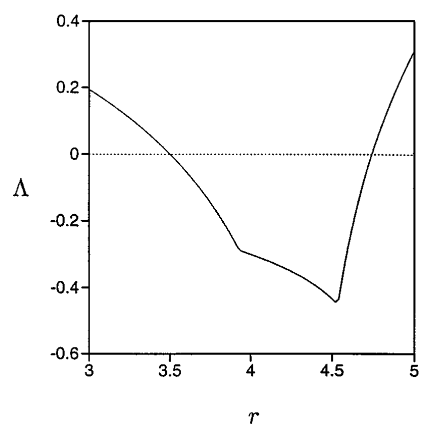

FIG. 3. The largest subsystem Lyapunov exponent Λ for the coupled map lattice Eq. (10) plotted as a function of the parameter r. Synchronism is achieved for r in the range $3.51 < r < 4.75$ over which Λ is negative.

where $\mathbf{z} \in \mathbf{R}^m$, $\mathbf{A}$ is an $m \times m$ constant matrix, $\mathbf{B}$ is a column vector, and $p(n)$ is a scalar control signal. Relating $p(n)$ to the current state of the system $\mathbf{z}(n)$ in a feedback fashion, $p(n) = -\mathbf{K}^T\mathbf{z}(n)$, Eq. (14) becomes

$$\mathbf{z}(n + 1) = (\mathbf{A} - \mathbf{B}\mathbf{K}^T)\mathbf{z}(n), \qquad (15)$$

which is in the same form as Eq. (13). The question for control theory is how to design a vector $\mathbf{K}$ such that $\mathbf{z} = \mathbf{0}$ becomes an attracting fixed point with any desired degree of stability. It can be shown [10] that, if the vectors $\mathbf{B}$, $\mathbf{AB}$, $\mathbf{A}^2\mathbf{B}, \ldots,$ and $\mathbf{A}^{m-1}\mathbf{B}$ are linearly independent, then for any given set of numbers μ_1, $\mu_2, \ldots, \mu_m$ one can find a $\mathbf{K}$ so that the matrix $\mathbf{A} - \mathbf{B}\mathbf{K}^T$ has this set of numbers as its eigenvalues. An equivalent way to express the above condition is to construct a controllability matrix $\mathbf{C} = (\mathbf{B} \mid \mathbf{AB} \mid \mathbf{A}^2\mathbf{B} \mid \cdots \mid \mathbf{A}^{m-1}\mathbf{B})$ and require $\mathrm{Det}(\mathbf{C}) \neq 0$.

For the coupled map lattice, the example of $m = 3$ with the set of parameters chosen above the determinant of the controllability matrix has the value of 0.08. This conforms with the finding in Fig. 3, showing that the synchronizing solution can be made stable by selecting suitable values of K_i. It is interesting to note that, if the coupling in Eq. (10) is uniform, i.e., $\epsilon_1 = \epsilon_2 = \epsilon_3$, then the controllability matrix has a zero determinant regardless of how $\mathbf{B}$ is chosen. This means that one will not be able to use a scalar transmitted signal to synchronize the receiver subsystem. But we argue that this is a very rare case in terms of all possible dynamical systems in the form of Eq. (10). A slight deviation from the uniform coupling situation will typically restore the controllability condition and enable again chaos synchronism by a single transmitted scalar signal.

In summary, we have considered the implementation of the Pecora-Carroll synchronization paradigm in high dimensional chaotic systems with two or more positive Lyapunov exponents. In particular, pertaining to the application to communication, we have addressed the issue of whether chaos synchronism is achievable in such systems by transmitting a single scalar signal. We give an affirmative answer to the question by proposing the use of a transmitted signal that is a linear combination of the original phase space variables. This approach provides adjustable parameters in the relative weight of these variables, and proves to be effective when applied to two typical examples of hyperchaos, one a differential equation system and the other a coupled map lattice.

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