Semiclassical Analysis of Energy Level Correlations for a Disordered Mesoscopic System

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A model for a mesoscopic system where hyperspherical impurities are placed at random in the interior of hypercubical billiard is studied. The disorder is characterized by the elastic mean free path l that ranges from the ballistic to the diffusive regime. The energy level correlator $K(s)$ and the form factor $\hat{K}(t)$ are calculated and characterized by analyzing the analytic structure of the dynamical zeta function of the corresponding classical system.

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The fingerprints of the underlying classical chaotic dynamics on the quantal behavior of the corresponding system are explored in the field of "quantum chaos" [1]. In particular, it was found that the level statistics of a variety of chaotic systems are described by random matrix theory (RMT) [2] over a certain energy domain [1]. Examples of such systems are the Sinai and stadium billiards. Beyond this energy domain there are deviations from the RMT universal behavior. For ballistic chaotic systems this time scale is of the order of the period of the shortest periodic orbits [3]. For disordered metallic grains it is the Thouless time, namely, the time required for the particle to diffuse across the system [4]. The corresponding energy scale is the Thouless energy.

In contrast with chaotic systems, the spectral statistics of generic integrable systems, such as the rectangular billiard, is Poissonian [5]. Since symmetry is crucial for such systems, it was considered for a long time impractical to realize them in the mesoscopic domain.

The advance in fabrication techniques enables us to manufacture mesoscopic systems that are relatively clean, namely, their elastic mean free path can be larger than the size of the system. Such systems were prepared with boundaries of different shapes and used to explore the difference between transport in chaotic and integrable systems [6]. One can prepare nanostructure devices so that their boundaries induce integrable classical dynamics if there is no disorder, e.g., when the boundary is a circle or a rectangle. Since the level statistics of generic integrable systems is Poissonian [5], one expects that the results of RMT cannot be applied directly for such systems. Yet, mesoscopic systems, even if prepared with a very high degree of cleanliness, inevitably contain some impurities. This motivates the study of a model in which the amount of disorder can be controlled, say, by fixing the number of impurities, and with the property that without disorder the classical dynamics is integrable. We mention here that such a system was realized experimentally using microwave cavities [7].

In earlier work, Altland and Gefen explored a related model where pointlike impurities were introduced into an integrable system [8]. In the present Letter a D-dimensional hypercubic billiard doped with rigid hyperspherical impurities is studied. The boundary conditions are assumed to be periodic, so that the geometry is of a D-dimensional torus. This model is a prototype system for integrable systems (where the motion in phase space is on tori) doped with impurities that scatter particles only within some finite range of their center. The trajectories that are not scattered by impurities are identical to those of the corresponding integrable system. The trajectories that are scattered by impurities may be chaotic, and are such if the impurities are rigid spheres. The calculations will be done in the framework of the semiclassical approximation. For the validity of this approximation it will be assumed that the energy is sufficiently high so that the wavelength of the particles is much smaller than the radius of the spheres. It will be assumed also that all the spheres have the same radius, that is, much smaller than the size of the system. The classical motion for this prototype system resembles the motion of a large variety of classical systems in angle-action variables. This similarity holds also in the framework of the semiclassical approximation [9] that will be used here.

The specific quantity that will be explored in detail is the dimensionless density-density correlator that is related to various physical quantities such as the conductivity. It is defined to be

$$K(s) = \Delta^2 \langle \rho(\epsilon)\rho(\epsilon + s\Delta)\rangle_\epsilon - 1\,, \qquad (1)$$

where $\rho(\epsilon)$ is the density of states at energy ϵ, Δ is the mean level spacing, and $\langle\cdots\rangle_\epsilon$ represents an averaging over some interval of the energy ϵ for a specific realization of disorder. An ensemble average over realizations of disorder is denoted hereafter by $\langle\cdots\rangle$. In what follows the ensemble averaged correlator $\langle K(s)\rangle$ will be calculated. A related function commonly used in this context is the dimensionless form factor $\hat{K}(t)$ which is the Fourier transform of $K(s)$, namely, $\hat{K}(t) = \int ds K(s) e^{-ist}$. The universal form of $\hat{K}(t)$ is especially simple in the unitary case. It is $\hat{K}(t) = \min(1, |t|/2\pi)$. Here we shall study the deviations from this universal behavior for the model described above.

Throughout this Letter dimensionless quantities will be used. In particular, the length of the billiard and the mass of the particle are chosen to be equal to unity; energy is

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measured in units of the mean level spacing ($s = \epsilon/\Delta$), and time is measured in units of $\hbar/\Delta$.

The density-density correlator may be represented as a sum of two terms [10,11]:

$$K(s) = K_P(s) + K_{\rm osc}(s)\,. \tag{2}$$

The first one, $K_P(s)$, is the smooth term given by the diagonal approximation [12] or calculated by diagrammatic perturbation theory [4]. In this approximation $K_P = K_P^{(i)} + K_P^{(c)}$ is a sum of two terms, where $K_P^{(c)}$ and $K_P^{(i)}$ are the contributions of orbits that are scattered or not scattered by impurities, respectively. It can be represented in the form

$$K_P(s) = -\frac{\beta}{4\pi^2}\frac{\partial^2}{\partial s^2}\ln[\mathcal{D}(s)]\,, \tag{3}$$

where β equals 2 (1) for the orthogonal (unitary) ensemble. $\mathcal{D}(s)$ is a spectral determinant normalized such that $\lim_{s\to 0} s^2\mathcal{D}(s) = 1$, and related to the dynamical zeta function $1/Z(z)$ via $\mathcal{D}(s) = Z(is)Z(-is)$. The zeros of $1/Z(z)$ are the eigenvalues of the Perron-Frobenius operator of the corresponding classical system [13]. They are associated with the decaying modes of the probability distribution of particles towards the ergodic distribution.

The second term, $K_{\rm osc}(s)$, cannot be calculated by usual perturbation theory. It was obtained for hyperbolic systems in the limit of large s retaining the nonperturbative terms of the correlator [10]. It turns out that $K_{\rm osc}(s)$ is also governed by the same classical spectral determinant. For in the unitary case it takes the form

$$K_{\rm osc}(s) = [\cos(2\pi s)/2\pi^2]\,\mathcal{D}(s)\,. \tag{4}$$

The result for the ensemble averaged perturbative part of the form factor will be presented now, and the dynamical zeta function $1/Z(z)$ will be identified. The calculations were performed using periodic orbit theory in the framework of the diagonal approximation, and the details will be published elsewhere [14]. The probability to find a long orbit that is not scattered by impurities decays exponentially with its length. The contribution of such orbits to the form factor was therefore found to decrease exponentially with time, namely,

$$\langle \hat{K}_P^{(i)}(t)\rangle = \beta e^{-2t/\tau}, \tag{5}$$

where τ is the elastic mean free time.

The contribution of orbits that are scattered by impurities to the density-density correlator is

$$\langle K_P^{(c)}(s)\rangle = \frac{\beta}{2\pi^2}\frac{\partial^2}{\partial s^2}\Re\sum_{\mathbf{m}}\{\ln[1 - \mathcal{G}_D(2\pi|\mathbf{m}|l, 1 - i\tau s)] + \mathcal{G}_D(2\pi|\mathbf{m}|l, 1 - i\tau s)\}\,, \tag{6}$$

where l is the elastic mean free path, $|\mathbf{m}|$ is the modulus of a D-dimensional integer vector, and

$$\mathcal{G}_D(x,y) = \frac{1}{\sqrt{x^2+y^2}}F\left(\frac{1}{2}, \frac{D}{2} - 1; \frac{D}{2}; \frac{x^2}{x^2+y^2}\right),$$

where $F(\alpha,\beta;\gamma;\delta)$ is the hypergeometric function. For example, for $D = 2$, $\mathcal{G}_2(x,y) = 1/\sqrt{x^2+y^2}$, while for the quasi-one-dimensional case $\mathcal{G}_1(x,y) = y/(x^2+y^2)$. Relation (3) together with the additive property of $K_P(s)$ imply that the dynamical zeta function has the form $Z(z) = Z_i(z)Z_c(z)$. The term $Z_i(z)$ is associated with orbits that are not scattered by impurities, and is given by

$$1/Z_i(z) = \left(\frac{2}{e}\right)^{4\pi/\tau} \times \exp\left\{-\frac{2\pi}{\tau}(2 - z\tau)[\ln(2 - z\tau) - 1]\right\}. \tag{7}$$

It is normalized such that $Z_i(0) = 1$. The second term $Z_c(z)$, coming from orbits which are scattered by impurities, is

$$1/Z_c(z) = \prod_{\mathbf{m}} B_{\mathbf{m}}[1 - \mathcal{G}_D(2\pi|\mathbf{m}|l, 1 - \tau z)] \times \exp[\mathcal{G}_D(2\pi|\mathbf{m}|l, 1 - \tau z)]\,. \tag{8}$$

Here $B_{\mathbf{m}}$ are regularization factors which are introduced to make the product converge, and to satisfy the normalization property $\lim_{z\to 0} zZ(z) = 1$. In the quasi-one-dimensional case the dynamical zeta function has the simple form

$$1/Z_c(z) = \mathcal{N} e^{\coth[(1-z\tau)/2l]/2l}\left[\frac{\sin\left(\sqrt{z\tau - z^2\tau^2}/2l\right)}{\sinh[(1 - z\tau/2l)]}\right]^2, \tag{9}$$

where $\mathcal{N}$ is a normalization constant. In the two-dimensional case it is straightforward to show that the infinite product (8) can be regularized by choosing the regularization factors $B_{\mathbf{m}}$ to satisfy

$$B_{\mathbf{m}}^{-1} = [1 - \mathcal{G}_2(2\pi l|\mathbf{m}|, 1)]\exp[\mathcal{G}_2(2\pi|\mathbf{m}|l, 1)]\,, \tag{10}$$

where $\mathbf{m} \neq \mathbf{0}$, while $B_{\mathbf{0}} = 1$. The analytic structure of $1/Z_c(z)$ is depicted in Fig. 1.

The behavior of the density-density correlator is controlled by the elastic mean free path l, and the elastic mean free time τ. The elastic mean free path measures the amount of disorder in the system. When $l > 1$, i.e.,

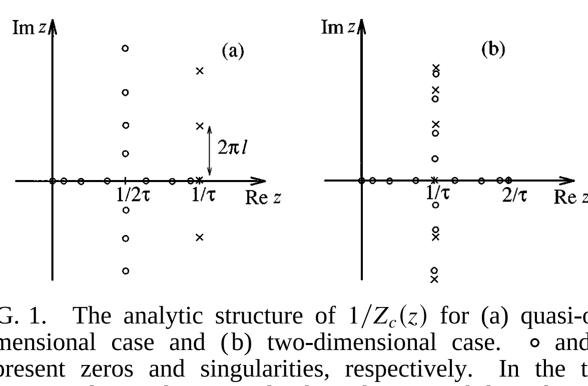

FIG. 1. The analytic structure of $1/Z_c(z)$ for (a) quasi-one-dimensional case and (b) two-dimensional case. ∘ and × represent zeros and singularities, respectively. In the two-dimensional case there are also brunch cuts with brunch points at the singularities.

when the elastic mean free path is larger than the system size, the dynamics is ballistic, while it is diffusive when $l \ll 1$ [8]. The elastic mean free time determines the classical time scales of the system, since $v = l/\tau$, where v is the dimensionless velocity. As $\tau \to 0$ other time scales effectively increase relative to τ, and the RMT universal behavior is recovered.

The nonuniversal features that appear for finite small values of τ decorate the RMT result of the form factor mainly in two regions: near the origin $t = 0$, and in the vicinity of the Heisenberg time $t = 2\pi$. They are appreciable over small intervals that scale with τ. Thus as $\tau \to 0$ these time domains shrink and the universal result is reached. The nature of the nonuniversal behavior of the system is determined by the elastic mean free path l. This behavior for a system with 2 degrees of freedom belonging to the unitary ensemble will now be described in some detail. The figures that will be presented were produced by fast Fourier transform of $K(s)$, where the product (8) required for $\mathcal{D}(s)$ is calculated numerically by including terms with $|m_i| \leq 1000$ $(i = 1, 2)$, and using the regularization (10).

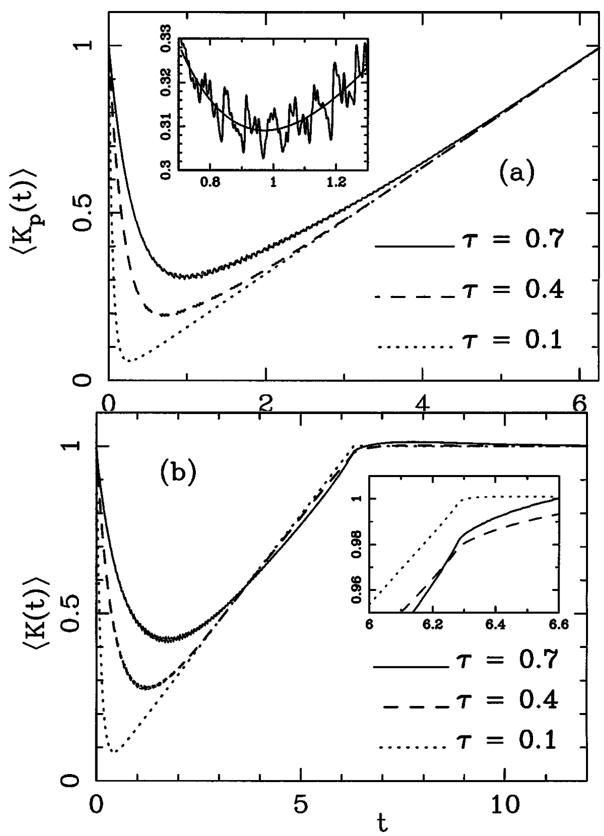

FIG. 2. (a) The perturbative part of the form factor for $l = 10$. The inset is a magnification of the region $0.7 \leq t \leq 1.3$ for the case $\tau = 0.7$ and $l = 10$. (b) The full form factor for the same parameters as in (a). The inset is a magnification of the vicinity of the Heisenberg time.

The typical structure of the form factor in the ballistic regime $l \gg 1$ is depicted in Fig. 2. For $t < \tau$ it is dominated by the contribution, $e^{-2t/\tau}$, of periodic orbits that are not scattered by impurities (5). Near the Heisenberg time $|t - 2\pi| < \tau$, the RMT singularity is smoothed out. The typical line shape of the form factor in this regime is characterized by a minimum in the vicinity of $t = \tau$. It is not clear if in this case (4) applies. Figure 2(b) should be therefore considered as a conjecture.

The form factor, in the intermediate regime $l = 0.5$, is depicted in Fig. 3. Here the contribution from periodic orbits that are not scattered by impurities is negligible, and the behavior is determined by orbits that do scatter. Near the origin, (see the inset in Fig. 3) the form factor exhibits a singular behavior. These singularities can be associated with short orbits that are scattered from a very small number of impurities, and therefore still preserve the topology of the orbits of the clean system. The amplitudes of the singularities decay exponentially as $e^{-t/\tau}$, since the probability for the existence of such orbits decreases exponentially with time. The behavior near the Heisenberg time is oscillatory with a period of order of the time of flight across the system τ_f.

Moving towards the diffusive regime where $l < 1/2\pi$, the nonuniversal features change their character as presented in Fig. 4, for $l = \sqrt{5}/6\pi \simeq 0.12$. The singular behavior near the origin almost disappears, and the oscillations near the Heisenberg time transform into an overall smooth curve. The situation does not change much when l is even smaller.

An understanding of the scenario described above can be obtained by analyzing the classical dynamical zeta function of the system, $1/Z(z)$. A crucial role is played by the singularities and the zeros of this function, since, up to constants, the perturbative part of the form factor, $\langle \hat{K}_P(t)\rangle$, and nonperturbative part, $\langle \hat{K}_{\rm osc}(t)\rangle$, are

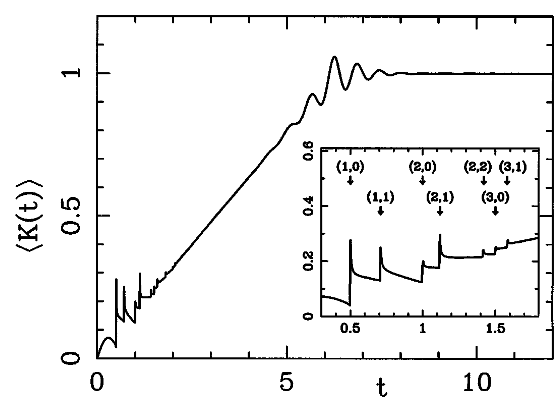

FIG. 3. The form factor for the parameters $l = 0.5$ and $\tau = 0.25$. The inset is a magnification of the domain $0.3 \leq t \leq 1.8$. The times indicated by arrows are the periods of the orbits which are not scattered by impurities. The pair (M_x, M_y) above each arrow represents the winding numbers, namely, the topology.

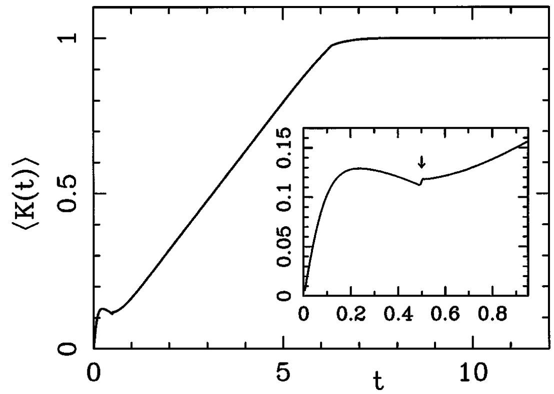

FIG. 4. The form factor for the parameters $l = 5^{1/2}/6\pi$ and $\tau = l/2$. The inset is for the time domain $0 < t < 0.9$.

the Fourier transforms (in $s = -iz$) of $\partial^2[\ln Z(z) + \ln Z(-z)]/\partial z^2$ and $Z(z)Z(-z)\cos(-i2\pi z)$, respectively. Therefore the zeros and singularities that are closest to the imaginary z axis dominate the behavior of the form factor.

The analysis of $1/Z(z)$ in the two dimensional case shows that its zeros associated with orbits which are scattered from impurities appear in the interval $[0, 2/\tau]$ with $\Im z = 0$, and on the line $\Re z = 1/\tau$ [see Fig. 1(b)]. When $l > 1/2\pi$ all of them except two zeros, one at the origin and the second at $z = 2/\tau$, lie on the line $\Re z = 1/\tau$. The singularities are located only along this line, $\Re z = 1/\tau$. The part of $1/Z(z)$ associated with orbits that are not scattered has a cut and a brunch point at $z = 2/\tau$. Since except the zero at the origin all these zeros and singularities scale as $1/\tau$, their contribution to $\langle \hat{K}(t) \rangle$ is expected to fall off exponentially with an exponent that scales as $1/\tau$. It therefore vanishes rapidly in the limit $\tau \rightarrow 0$, and only the RMT term associated with the zero at the origin survives.

In the ballistic regime $l \gg 1$, the contribution from all zeros and singularities that lie at $\Re z = 1/\tau$ is negligible. Thus for $|t| < \tau$, $\langle \hat{K}(t) \rangle$ is dominated by the contribution (5) of periodic orbits that are not scattered by impurities.

As l becomes of order unity, the contribution of nonscattered orbits becomes negligible and the behavior is governed by orbits which do scatter. If also $l > 1/2\pi$, then all the zeros z_n of $1/Z(z)$, except those at the origin and at $2/\tau$, are complex and located along the line $\Re z = 1/\tau$. The nonuniversal features are therefore oscillatory, because the contribution of a zero, z_n, to $\langle \hat{K}_P(t) \rangle$ and $\langle \hat{K}_{\rm osc}(t) \rangle$ is proportional to $\exp(-z_n t)$ and $\exp(-z_n |t - 2\pi|)$, respectively [11].

Zeros of $1/Z(z)$ start to appear on the real z axis, when l becomes smaller than $1/2\pi$. Some of them dominate the behavior of the form factor, since they are closer to the imaginary z axis than the complex zeros. In particular, the nonuniversal features of the form factor become nonoscillatory. This change of behavior is demonstrated in Figs. 3 and 4.

The diffusive limit is reached when $l \ll 1/2\pi$. The real zeros in this limit concentrate near the origin. By summing over them one recovers the Altshuler and Shklovskii result for the perturbative part of the form factor $\langle \hat{K}_P(t) \rangle \approx \beta\tau/4\pi^2 l^2$ $(D = 2)$ [4], which applies for times smaller than $\tau_c/4\pi^2$, where $\tau_c = D\tau/l^2$ is the Thouless time.

The form factor of a disordered system which is integrable in the ballistic limit was calculated. Its nonuniversal features near the origin and in the vicinity of the Heisenberg time were characterized. These are determined by properties of the classical dynamics over short time scales.

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[1] F. Haake, *Quantum Chaos* (Springer, New York, 1991).
[2] M. L. Mehta, *Random Matrices* (Academic Press, New York, 1991).
[3] M. V. Berry, Proc. R. Soc. London A **400**, 229 (1985).
[4] B. L. Altshuler and B. I. Shklovskii, JETP **64**, 127 (1986).
[5] M. V. Berry and M. Tabor, Proc. R. Soc. London A **356**, 375 (1977).
[6] C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, Phys. Rev. Lett. **69**, 509 (1992); A. M. Chang, H. U. Baranger, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **73**, 2111 (1994); T. M. Fromhold, L. Eaves, F. W. Sheard, T. J. Foster, M. L. Leadbeater, and P. C. Main, Physica (Amsterdam) **201B**, 367 (1994).
[7] A. Kudrolli, V. Kidambi, and S. Sridhar, Phys. Rev. Lett. **75**, 822 (1995); J. Stein and H.-J. Stöckman, Phys. Rev. Lett. **68**, 2867 (1992); S. Sridhar and E. J. Heller, Phys. Rev. A **46**, 1728 (1992).
[8] A. Altland and Y. Gefen, Phys. Rev. Lett. **71**, 3339 (1993); Phys. Rev. B **51**, 10 671 (1995).
[9] M. V. Berry and M. Tabor, J. Phys. A **10**, 371 (1977).
[10] A. V. Andreev and B. L. Altshuler, Phys. Rev. Lett. **75**, 902 (1995).
[11] O. Agam, B. L. Altshuler, and A. V. Andreev (to be published).
[12] M. V. Berry, in *Chaos and Quantum Physics*, Proceedings of Les-Houches Summer School, Session LII, edited by M. J. Giannoni, A. Voros, and J. Zinn-Justin (Elsevier, Amsterdam, 1991).
[13] D. Ruelle, *Statistical Mechanics, Thermodynamic Formalism* (Addison-Wesley, Reading, MA, 1978).
[14] O. Agam and S. Fishman (to be published).