Shape of the Tail of a Two-Dimensional Sandpile

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We study the shape of the tail of a heap of granular material. A simple theoretical argument shows that the tail adds a logarithmic correction to the slope given by the angle of repose. This expression is in good agreement with experiments. We present a cellular automaton that contains gravity, dissipation, and surface roughness and its simulation also gives the predicted shape. [S0031-9007(96)00515-7]

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Many typical phenomena observed in granular materials have been studied in the last years [1–3]. In order to understand these phenomena simple models as well as computer intensive simulation techniques [4] have been useful. In this Letter we discuss the shape of static heaps of granular material.

Sandpiles are almost perfect cones with a well defined angle of repose which depends on gravity and on the characteristics of the material. Watching carefully, however, one notices the existence of curved tails at the bottom of the heaps to which not much attention has been paid in the literature. Here we study in detail the shape of these tails.

We will present a simple experiment to obtain the profile of the tails in two dimensions. Then, we deduce an analytical expression for their shape which agrees with our experimental profiles. This expression has two parameters that depend on the material properties. Finally, we test the theory with data obtained from computer simulations using cellular automata (CA). Several CA models have been proposed in the past to study the behavior of granular matter [5–8]. They include dissipation, which is the most important ingredient to capture the peculiarities of granular media. One such automaton is a lattice gas formulation by Peng and Herrmann [8] with rest particles and inelastic collision rules. Using this model we have simulated two-dimensional heaps in a quasistatic regime and compared the resulting profiles with those predicted by our theory.

We built sandpiles in an easy and inexpensive experiment. Grains were poured at a slow rate of about 10 particles/sec from the top into the center of a rectangular cell made of two parallel vertical glass plates of size 30×25 cm^2 separated by a fixed distance of 2 mm. As granular materials we used lead spheres, sugar, and polenta. The grain diameter was 2 mm in the first case and about 0.5 mm in the other two cases. In Fig. 1(a) we see a digitized image of the experiment showing the essentially two-dimensional heaps that we obtain at different times. We have studied the profiles of the heaps by recording digitized pictures taken by a VHS video camera at different stages of the evolution. The resolution of digitized images was 40 pixels/cm.

The heaps are grown in steady state, in the sense that only a few grains move simultaneously, being in the constant velocity regime [9]. In this regime particles do not accumulate at the top of the heap and the formation of big avalanches which can disturb the formation of smooth profiles is not present. Therefore profiles do not depend on the flow rate at the top. They only depend on the type of grains used and gravity.

In Fig. 1(a) we see the typical situation of two flat surfaces that define the angle of repose. We will, however, focus our attention on the small curved tails that can be observed at the base of the heaps and obtain the analytical expression of their shape using a simple argument.

Taking pictures at different times we discovered that it is possible to superpose the profiles of the tails by a simple horizontal shift [see Fig. 1(a)]. It seems that the heap grows by putting layers of particles over one another. Each individual layer grows upwards from the bottom to the top by stopping particles which are moving down along the surface of the heap.

At the top of each incomplete layer there is a kink [kinks are marked by arrows in Fig. 1(b)]. (In the case of irregularly shaped particles or particles of different size one has asperities of different size on the surface of the heap, and the larger ones effectively act as the kinks discussed here.) The presence of such kinks reduces the slope of the surface away from the angle of repose of the material. Let us describe the surface by $h(x)$ where h is the height and x the corresponding horizontal displacement. We choose the origin at the center of the base of the heap, i.e., $h(0) = h_m$ where h_m is the height of the heap at the top, and consider only $x > 0$ since the heap is symmetric with respect to the origin. In the absence of kinks one would have $h = h_m - \gamma x$ where $\gamma = \tan\theta$ and θ is the angle of repose. The presence of each kink increases this ideal value of x by a certain value l which typically is of the size of a grain. If ρ is the number of kinks per unit length in the vertical direction, we can express the slope of the surface as a function of ρ as

$$\frac{dh}{dx} = -\frac{\gamma}{1 + l\gamma\rho}\,. \qquad (1)$$

Particles falling down along the surface collide with the kinks and can be accumulated on top of them with some rate r or continue to move downwards to the next layer.

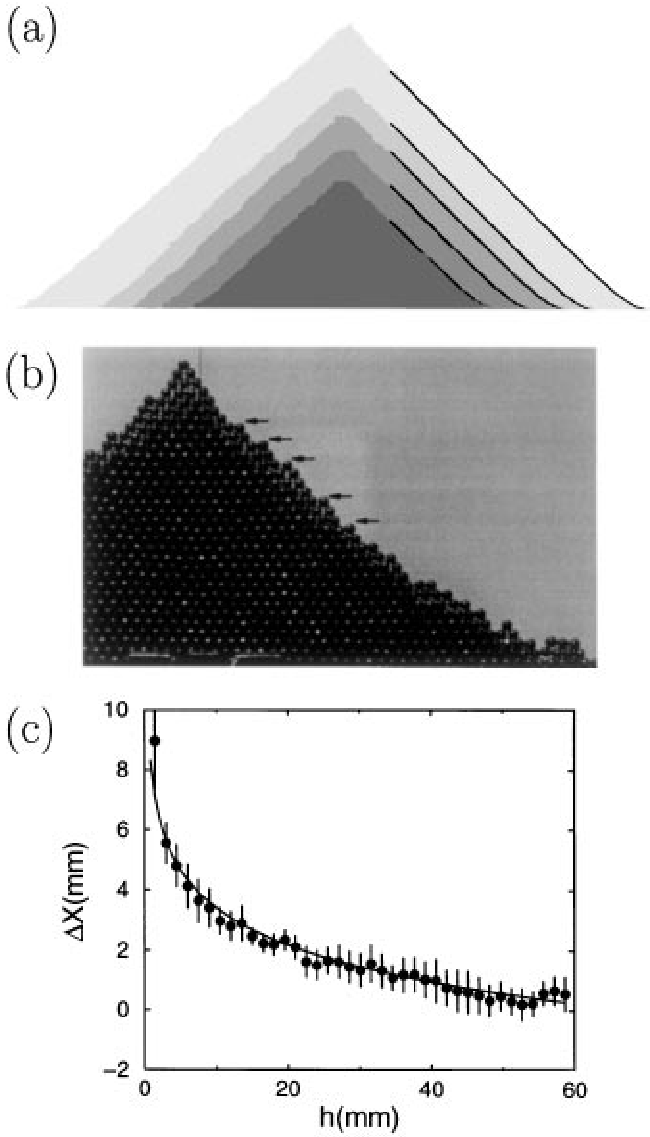

FIG. 1. (a) Digitized image of a heap of the polenta. The diameter of the grains is about 0.5 mm. The height of the heap is 16 cm. Different gray levels show the pile at different stages of growth. The superposed continuous lines in both figures are fits obtained from Eq. (3) by taking the values $\gamma = 0.98$ and $l_e = 1.5$ mm. (b) Tail of a pile made of lead spheres with a diameter of 2 mm. One can observe parallel layers terminating in horizontal kinks (marked by arrows). (c) Deviation of the shape from the straight profile for the same material. Data are averages obtained from 10 different samples. Error bars represent the standard deviations of the averaged values. Continuous line represents the same fit of (a).

r depends on l and on the properties of the grains (shape, roughness, coefficient of restitution). Let us call $\Phi(h)$ the flux of particles pulled down by gravity along the surface. Since the experiments were performed in steady state, one has the relation $d\Phi/dh = r\Phi\rho$. The fact that the heap grows by a shift of the profile in the horizontal direction means that the aggregation rate of particles is independent of h. Since the number of particles aggregated per vertical unit length is the variation of the vertical flux, we have $\partial\Phi/\partial h = B$, where B is a constant and since $\Phi(0) = 0$ we obtain $\Phi = Bh$. Putting everything together the slope of the surface as a function of ρ is given by

$$\frac{dh}{dx} = -\frac{\gamma}{1 + l\gamma/rh}\,. \tag{2}$$

After integration we obtain the final expression

$$x = \frac{h_m - h}{\gamma} + l_e \ln \frac{h_m}{h}\,, \tag{3}$$

which shows a logarithmic correction to the angle of repose. $\gamma = \tan\theta$ is the classical dynamic friction coefficient and $l_e \equiv l/r$ describes the typical extra horizontal displacement a particle must undergo before it sticks. Both constants characterize the granular material. In Fig. 1(c) we show the average over 10 different profiles representing the deviation from the straight profile. Error bars are the standard deviations for each value of h. The surface fit is for $\gamma = 0.98$ and $l_e = 1.5$ mm. It is very interesting to note that for the three materials used here we find consistence with $r \approx 1/3$.

In the following we will check our formula using numerical simulations. We consider a lattice gas automaton (LGA) at integer time steps $t = 0, 1, 2, \ldots$ with particles located at the sites of a two-dimensional triangular lattice of size L. Gravity acts downward in the vertical direction and forms an angle of 30° with the closest lattice axis. At each site there are seven bit variables which refer to the velocities v_i $(i = 0, 1, 2, \ldots, 6)$. Here v_i $(i = 1, \ldots, 6)$ are the nearest neighboring (NN) lattice vectors and $v_0 = 0$ refers to the rest state (zero velocity). Each state can be either occupied by a single particle or empty. Therefore, the number of particles per site has a maximal value of seven and a minimal value of zero.

The time evolution of the LGA consists of a collision step and a propagation step. In the collision step particles can change their velocities due to collisions and in the propagation step particles move in the direction of their velocities to the NN sites where they collide again. The system is updated in parallel. Only the collisions specified in Fig. 2 can deviate the trajectories of particles from straight lines with probabilities depending on the dissipation parameter p. If p is not zero, energy can be dissipated during the collision. This is a crucial property of granular materials and yields among others an instability towards cluster formation [10].

The two collisions shown on the lower part of Fig. 2 temporarily allow more than one particle on a site. However, immediately after the collision step, the extra rest particles randomly hop to NN sites until they find a site with no rest particle, and there they stop.

We incorporate the driving force, namely, gravity g, by setting a rest particle into motion with probability $g/2$

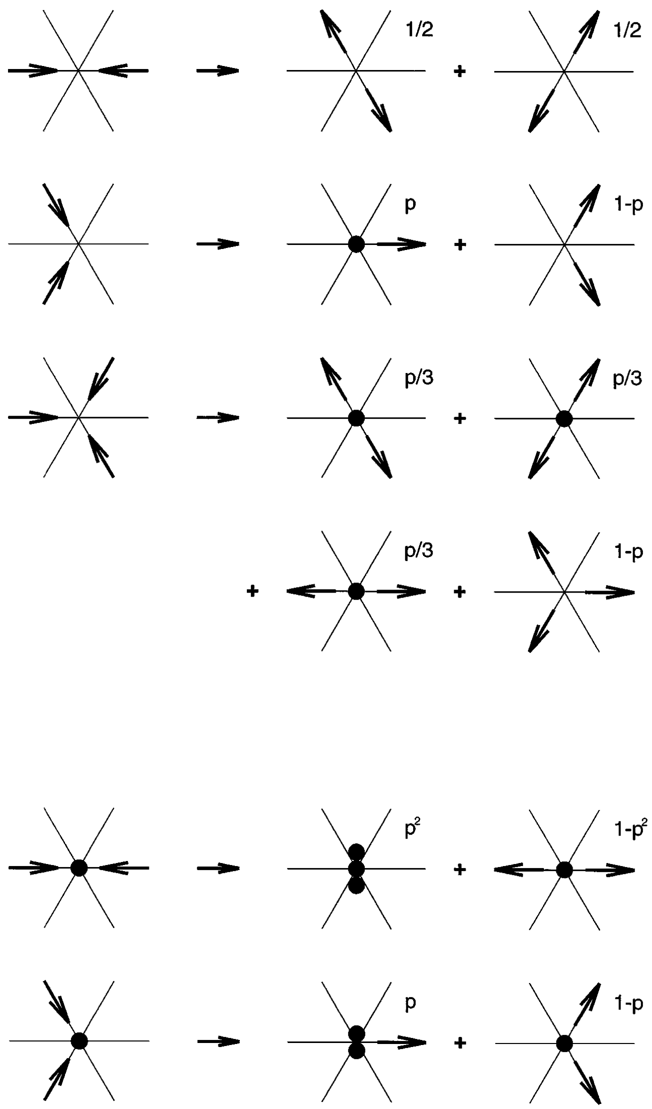

FIG. 2. Collision rules of the cellular automaton. Arrows represent moving particles and full dots stand for rest particles. The number next to each configuration is the probability for that transition.

along one of the two lattice directions which form an angle of 30° with the direction of gravity, however, only if the site below is empty at that time. Rest particles that are already on the heap can only be accelerated by gravity if at least one of the two NN sites in the direction of gravity does not yet belong to the heap.

During the evolution, we add particles at a fixed rate from one site at the top to the system. Gravity moves these particles down until they collide with the hard wall at the bottom. A particle colliding with the bottom is either reflected with probability $1 - p$ in the specular direction or stopped with probability p losing its momentum. In the second case the resulting rest particle belongs to the growing heap, and we label that particle in order to store the information that it is sustained by the bottom plate. Every particle colliding with these particles loses its momentum and is aggregated to the heap after the redistribution process described above has taken place.

We perform simulations for systems of size $L = 512$ typically iterating 3×10^5 time steps. Most of the computer work was performed on a CM-5. As in the experiment we add few particles (one every 8 time steps) to be in a quasistatic regime. In that case the profile does not depend on the history of the system. Resulting configurations are shown in Fig. 3(a).

For all parameter values the profiles have an angle of repose of 60° corresponding to $\gamma = \sqrt{3}$ which is determined by the geometry of the underlying lattice. The tail shows the presence of kinks which reduce the slope. The different gray levels correspond to different time steps. One notes that as in the experiment the surfaces of the heap are just horizontally displaced in time.

For this model the parameters in Eq. (3) are easily determined. On the one hand, we use as unit length the distance between NN sites on the underlying lattice which

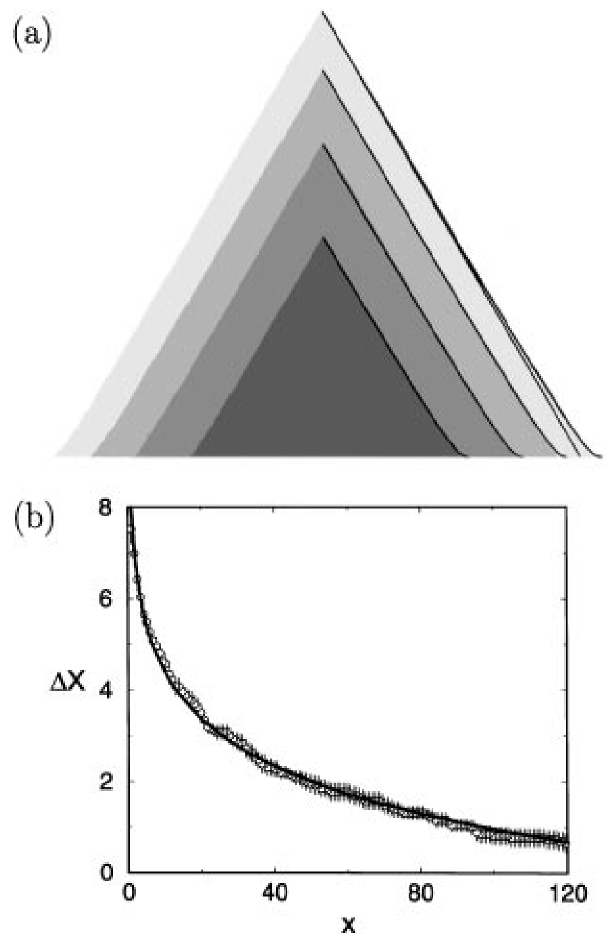

FIG. 3. (a) Heap of 80 000 particles obtained in a quasistatic regime adding one particle every 8 time steps. The different gray levels visualize the growth after every 160 000 steps of evolution. Continuous lines are fits obtained from Eq. (3). (b) Deviations from the straight line of averaged profiles obtained from 16 independent samples. The fit is the same as in (a).

we choose to be unity giving $l = 1$. On the other hand, $r = 1/3$ because every particle colliding with a kink is aggregated to the heap and then redistributed randomly to one of the three empty NN sites. Only one of these site particles will aggregate on top of a layer.

In Fig. 3(b) we have plotted an average of 16 independent profiles in the same manner as in Fig. 1(b). The continuous line is the shape resulting from Eq. (3) with $\gamma = \sqrt{3}$ and $l_e = 3$. Our formula is in excellent quantitative agreement with the simulation.

In order to include into the CA the sticking of particles typical for wet sand we introduce a new parameter, namely, the probability η that one particle stopped and aggregated after a collision with the heap is "blocked." Gravity can only move a blocked particle if none of the two NN sites below belongs to the heap and then the particle is no longer blocked. Introducing these rules the slope of the surfaces of the heaps can be larger than 60° because each blocked particle can support rest particles.

Averaged profiles for $\eta = 0.002$ and 0.004 are shown in Fig. 4. We observe that now there is no well defined angle of repose as in the case of dry sand. The profile can be calculated using very similar arguments to those used before.

Let ρ_0 be the vertical density of blocked particles on the surface. The relation between the slope of the profile and ρ_0 is then

$$\frac{dh}{dx} = -\frac{\sqrt{3}}{1 + \sqrt{3}\,(\rho - \rho_0)}\,, \tag{4}$$

because the presence of each blocked particle on the surface reduces the value of x by $l = 1$.

It is easy to find a relation between ρ and h. On the one hand, $\rho_0 \propto \eta\Phi$. On the other hand, $\Phi \propto h$ due to the translational invariance which we have observed experimentally so that $\rho_0 = c'\eta h$. One therefore obtains

$$\frac{dh}{dx} = -\frac{\sqrt{3}}{1 + \sqrt{3}\,(1/rh - c'\eta h)}\,, \tag{5}$$

which after integration gives

$$x = \frac{h_m - h}{\sqrt{3}}\,[1 - c(h + h_m)] + r^{-1}\ln\frac{h_m}{h}\,, \tag{6}$$

where $c = (\sqrt{3}/2)c'\eta$ is a constant. In Fig. 4 we also show the fits obtained from this expression using $c' = 0.39$ and $r = 0.35$. We have checked that $\rho_0 \propto h$ in agreement with the argument for $h < 100$. If h becomes larger than 100, the density of blocked particles saturates to $\rho_0 \approx 0.28$.

We have studied experimentally, theoretically, and numerically with a cellular automaton the tail of sandpiles. The theoretical argument giving the shape of the tail is based on the experimentally observed translational invariance and uses mass conservation. One finds that the tail constitutes a logarithmic correction to the naive straight slope given by the angle of repose. The analytic expression fits very well to the experimental shape and to the two-dimensional heaps obtained with a cellular automaton.

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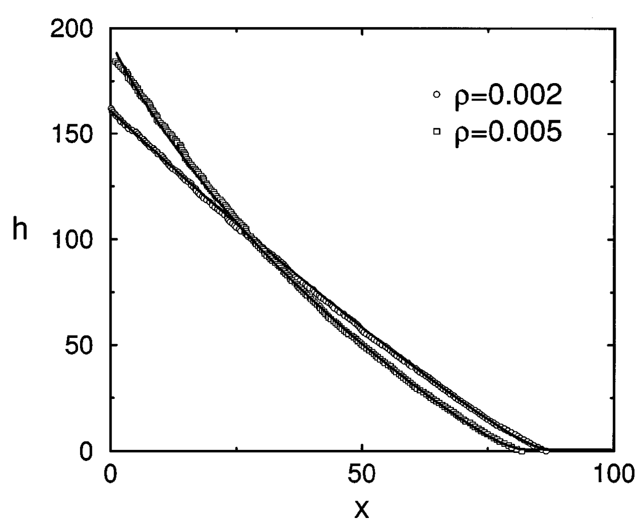

FIG. 4. Shapes obtained from 16 independent samples for a roughness parameter $\eta = 0.002$ (circles) and $\eta = 0.005$ (squares). Errors are of the same size as symbols. Continuous lines are fits obtained from Eq. (6) with $c' = 0.39$ and $r = 0.35$.

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