Interaction-Induced Delocalization of Two Particles in a Random Potential: Scaling Properties

Felix von Oppen, Tilo Wettig, and Jochen Müller
Max-Planck-Institut für Kernphysik, 69117 Heidelberg, Germany
(Received 5 April 1995; revised manuscript received 11 July 1995)

The localization length ξ_2 for coherent propagation of two interacting particles in a random potential is studied using a novel and efficient numerical method. We find that the enhancement of ξ_2 over the one-particle localization length ξ_1 satisfies the scaling relation $\xi_2/\xi_1 = f(u/\Delta_\xi)$, where u is the interaction strength and Δ_ξ the level spacing of a wire of length ξ_1. The scaling function f is linear over the investigated parameter range. This implies that ξ_2 increases faster with u than previously predicted. We also study a novel mapping of the problem to a banded-random-matrix model.

PACS numbers: 72.15.Rn, 71.30.+h

While much is known about the localization properties of one particle moving in a random potential [1], there are few secured results about localization in the presence of interactions between the particles [2]. In view of the complexity of the interplay of disorder and interaction, Dorokhov [3] and, very recently, Shepelyansky [4] approached this problem by studying a simple special case—two interacting particles in a random one-dimensional potential—and predicted that the interaction can lead to a significant delocalization of the pair. A possible realization of this system are excitons in a disordered semiconductor [3]. Furthermore, understanding the localization properties of two particles in a random potential may lead to new insights into the role of interactions in the Anderson insulator.

Shepelyansky [4] considered the motion of two particles interacting by a short-range interaction in a random potential. Whenever the two particles are localized far apart compared to the one-particle localization length, the effect of the interaction is only exponentially small. However, an interesting effect occurs when the two particles are localized within about one one-particle localization length of each other. In this case, Shepelyansky constructed an approximate mapping of the problem to a banded-random-matrix model. Studying this model numerically, he predicted that *independently of the statistics of the particles and the sign of the interaction* there is the possibility of coherent propagation of the two particles over distances ξ_2 much larger than the one-particle localization length ξ_1 [4]. He found that

$$\frac{\xi_2}{\xi_1} \approx \frac{\xi_1}{32}\left(\frac{u}{t}\right)^2, \tag{1}$$

where u denotes the interaction strength and t is the hopping matrix element. ξ_1 is measured in units of the lattice constant. Subsequently, Imry [5] has given a Thouless-type scaling argument supporting and generalizing this result, and Frahm *et al.* [6] have studied the problem numerically using a transfer-matrix technique, finding a slower increase of ξ_2 with ξ_1 than predicted by Eq. (1). Related results had been found earlier by Dorokhov [3] for the propagation of two harmonically bound particles in a random potential.

In this paper, we present a novel and efficient numerical technique to compute the two-particle localization length ξ_2 directly from a microscopic model. This method allows us to obtain accurate results over a wide range of parameters for both bosons and fermions. Our main result is that ξ_2 obeys the scaling relation $\xi_2/\xi_1 = f(u/\Delta_\xi)$, where Δ_ξ is the single-particle level spacing of a wire of length ξ_1. We conjecture, based on our numerical results, that the exact scaling function f is linear at the center of the band. While our results qualitatively confirm the prediction that a short-range interaction can lead to coherent propagation of the pair over distances much larger than the one-particle localization length, this scaling relation is inconsistent with the original prediction, Eq. (1). It is an important consequence of our results that the enhancement of ξ_2 sets in for *weaker* interactions than previously predicted. We also derive and study a novel mapping of the problem to a banded-random-matrix model. A combination of scaling with this banded-random-matrix model suggests the possibility that the validity of our principal results extends to quasi-one-dimensional wires.

Our starting point is the Anderson Hamiltonian H_0 for two spinless particles in a one-dimensional random potential with an additional Hubbard-type interaction U,

$$H = t\sum_{n,m}\{|n,m\rangle\langle n+1,m| + |n,m\rangle\langle n,m+1| + \text{H.c.}\} + \sum_{n,m}|n,m\rangle(V_n + V_m)\langle n,m| + U\,. \tag{2}$$

The random site energies V_n are drawn uniformly from the interval $[-W/2, W/2]$. The hopping matrix element t will be set to unity in the following. We parametrize the disorder by the one-particle localization length [1] $\xi_1 = 105(t/W)^2$ in the absence of the interaction. For bosons, we choose an on-site interaction with matrix elements $\langle n,m|U|n',m'\rangle = u\delta_{n,n'}\delta_{m,m'}\delta_{n,m}$, for spinless fermions a nearest-neighbor interaction with matrix elements $\langle n,m|U|n',m'\rangle = u[\delta_{n,m+1} + \delta_{n,m-1}]\delta_{n,n'}\delta_{m,m'}$. In the following our numerical method is described for

bosons. The extension to fermions is straightforward. To study the two-particle localization properties of the Hamiltonian (2) we focus on the matrix elements of the two-particle Green function

$$G = (E - H_0 - U)^{-1} \tag{3}$$

between doubly occupied sites $|n,n\rangle$. We define the two-particle localization length ξ_2 for coherent transport by the exponential decrease with distance of these matrix elements,

$$\frac{1}{\xi_2} = -\lim_{|n-m|\to\infty} \frac{1}{|n-m|} \ln|\langle n,n|G|m,m\rangle| . \tag{4}$$

This identification is certainly reasonable as long as the doubly occupied sites do not effectively decouple from the remaining Hilbert space because of the interaction, i.e., as long as the interaction strength u is smaller than or of the order of the hopping matrix element t. Our numerical approach exploits the observation that, for the interaction U, a closed equation can be derived for these matrix elements. This reduces the dimension of the relevant Hilbert space from N^2 to N (with N the number of sites) which enables us to study systems with up to 1000 sites. The Dyson equation for the two-particle Green function is

$$G = G_0 + G_0 U G , \tag{5}$$

where $G_0 = (E - H_0)^{-1}$ denotes the Green function in the absence of the interaction. The interaction can be written as $U = uP$, where P denotes the projector onto the doubly occupied sites, $P|n,m\rangle = \delta_{n,m}|n,m\rangle$. Hence, multiplying the Dyson equation by P on both sides and writing $U = uP^2$, one obtains a closed equation for the two-particle Green function projected onto doubly occupied sites,

$$\tilde{G} = \tilde{G}_0 + u\tilde{G}_0\tilde{G} . \tag{6}$$

Here we defined $\tilde{G} = PGP$ and $\tilde{G}_0 = PG_0P$. Solving this equation for $\tilde{G}$ one has

$$\tilde{G} = \frac{\tilde{G}_0}{u} \frac{1}{1/u - \tilde{G}_0} . \tag{7}$$

In the site basis, the unperturbed Green function $\tilde{G}_0$ is a banded matrix whose matrix elements decrease exponentially with distance on the scale $\xi_1/2$. Therefore, we compute ξ_2 using only the second factor in (7) from which any long-range behavior of G must arise. This is very useful for numerical purposes because this factor can be interpreted as the Green function of the "Hamiltonian" $\tilde{G}_0$ at "energy" $1/u$. This enables us to employ the efficient recursive Green-function method for banded Hamiltonian matrices [7] to find the two-particle localization length. We obtain the exact $\tilde{G}_0$ [8],

$$\langle n,n|\tilde{G}_0(E)\,|m,m\rangle = \sum_{i,j} \frac{\phi_i(n)\phi_j(n)\phi_i^*(m)\phi_j^*(m)}{E - E_i - E_j} , \tag{8}$$

by solving the Anderson model in the absence of the interaction. Here the ϕ_i are the exact single-particle wave functions. Clearly, our method of computing ξ_2 is accurate whenever the enhancement factor ξ_2/ξ_1 is sufficiently large. Deviations from the exact ξ_2 arise for small u and small ξ_1 where the enhancement is weak. For small u this can be easily seen because the second factor in (7) gives $\lim_{u\to 0} \xi_2 = 0$, while the exact limit is $\xi_1/2$.

We have studied the two-particle localization length ξ_2 for both fermions and bosons for one-particle localization lengths $4.2 \le \xi_1 \le 105$ and interaction strengths $0 \le u \le 1$. We find that it is sufficient to use systems with $N = 500$ sites except for the two largest values of ξ_1 where we used $N = 1000$. For each value of ξ_1 we averaged over 50 realizations of the disorder. In Fig. 1 we have plotted ξ_2/ξ_1 as a function of $u\xi_1/t$ at the center of the band ($E = 0$). We have included data for ten values of u for each of five values [9] of ξ_1. The observed scaling behavior

$$\frac{\xi_2}{\xi_1} = \tilde{f}(u\xi_1/t) \tag{9}$$

is the central result of this paper. While the data in Fig. 1 are for $E = 0$, we find that the same scaling behavior holds also away from the center of the band [10].

This scaling behavior implies that the scale for the interaction strength u is the energy t/ξ_1 which can be interpreted as the single-particle level spacing $\Delta_\xi = \pi t/\xi_1$ of

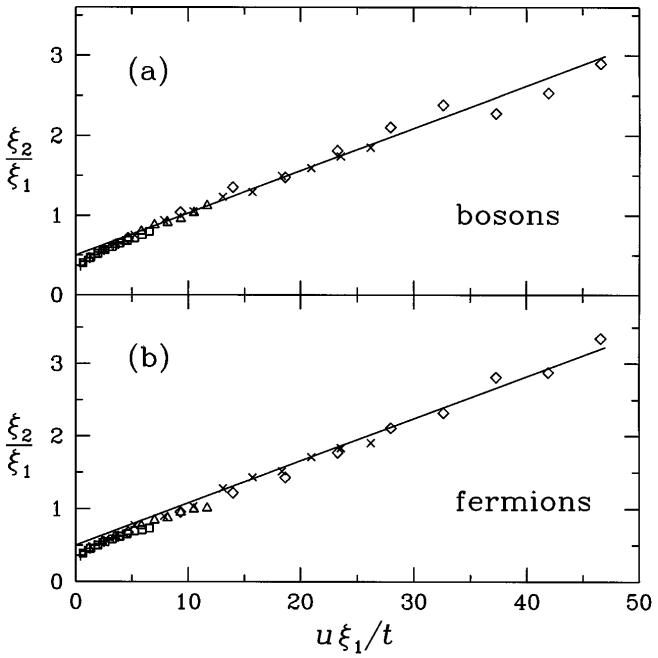

FIG. 1. Scaling plot $\xi_2/\xi_1 = \tilde{f}(u\xi_1/t)$ for the two-particle localization length ξ_2 of (a) bosons and (b) fermions as a function of interaction strength u and one-particle localization length ξ_1. Ten values of u are included for each of the five values of disorder $W = 5$ (plusses), $W = 4$ (squares), $W = 3$ (triangles), $W = 2$ (crosses), and $W = 1.5$ (diamonds). The full lines show that the linear behavior for large $u\xi_1/t$ extrapolates to $\xi_2/\xi_1 = 1/2$ for $u \to 0$. The deviation from linear behavior for small $u\xi_1/t$ is most likely an artifact of our numerical method.

a wire of length ξ_1. The scaling function in Fig. 1 can be fit well by a straight line for sufficiently large u/Δ_ξ while there are deviations from linear behavior for small values of the scaling variable. It is natural to suppose that these deviations from linear behavior are an artifact of disregarding the first factor in Eq. (7) in computing ξ_2. In fact, as shown by the full line in Fig. 1, the linear behavior for large u/Δ_ξ extrapolates to $\xi_2/\xi_1 = 1/2$ for $u/\Delta_\xi \to 0$ as expected for the exact two-particle localization length ξ_2 computed from the full expression for $\tilde{G}$. Hence, one may conjecture that the exact scaling relation at the center of the band has the form

$$\frac{\xi_2}{\xi_1} = \frac{1}{2} + C\frac{|u|}{\Delta_\xi}, \tag{10}$$

where $C \approx 0.17$ for bosons and $C \approx 0.18$ for fermions [11]. We used the fact that at the center of the band there is an exact symmetry between attractive and repulsive interactions so that ξ_2 depends only on the absolute value of u. Note that while this result is presumably valid for arbitrary values of ξ_1, one expects the scaling relation to break down for large u where u/t should become a relevant parameter due to density-of-state effects.

Shepelyansky's original prediction, Eq. (1), is *not* consistent with the scaling relation (9). E.g., at the center of the band our results show that while ξ_2 depends quadratically on ξ_1 as previously predicted, it exhibits an unexpected linear (instead of quadratic) dependence on the interaction strength u. More generally, the scaling (9) implies that the enhancement effect sets in for weaker interactions $u \sim t/\xi_1$ [compared to $u \sim t/\xi_1^{1/2}$ according to Eq. (1)] than originally predicted. This result is surprising in view of the following estimate. It may be argued [5] that ξ_2 should deviate from ξ_1 once the two-particle product states $|\phi_i, \phi_j\rangle$ of (unperturbed) energy $E_{i,j}$ are strongly mixed by the interaction U. According to perturbation theory, strong mixing occurs when $\langle\phi_1, \phi_2|U|\phi_3, \phi_4\rangle/(E_{1,2} - E_{3,4})$ is of order unity. Each $|\phi_i, \phi_j\rangle$ is typically coupled appreciably to ξ_1^2 states. The corresponding matrix element can be estimated [4,5] as $u/\xi_1^{3/2}$ and the energy denominator as t/ξ_1^2. Thus, according to this estimate, strong mixing occurs once $u \sim t/\xi_1^{1/2}$. Since this interaction strength is large compared to Δ_ξ, a comparison with our result would suggest that, surprisingly, strong mixing of the two-particle product states is not necessary for the enhancement of the two-particle localization length.

Originally, Shepelyansky [4] approached the problem by an approximate mapping to a banded-random-matrix model. We have also investigated an alternative random-matrix model which is suggested by Eq. (7) due to the band-matrix structure of $\tilde{G}_0$. An extension of this banded-random-matrix model will be applied below to study two interacting particles in a quasi-one-dimensional wire. In contrast to ordinary banded random matrices we find that the matrix elements g of $\tilde{G}_0(E = 0)$ have a Cauchy distribution, $P(g) = (\Gamma/\pi)/(\Gamma^2 + g^2)$. To obtain this distribution function we argue as follows. For definiteness, consider a diagonal matrix element of $\tilde{G}_0$. Because of the localized nature of the wave functions there are of the order of ξ_1^2 terms in the sum in (8). Furthermore, normalization implies that the wave functions are of order $1/\sqrt{\xi_1}$ within a region of size ξ_1. Hence, $\langle n, n|\tilde{G}_0|n, n\rangle \sim (1/\xi_1^2)\sum_{k=1}^{\xi_1^2}(1/x_k)$, where $x_k \sim -E_i - E_j$ is a random variable in the range $-4t \lesssim x_k \lesssim 4t$. Thus, the matrix elements of the Green function are given by averages over random variables whose second moments diverge. Neglecting correlations between the x_k, the central-limit theorem implies for sufficiently large ξ_1 that the diagonal matrix elements have a Cauchy distribution of width $\Gamma \sim 1/t$ [12]. The same argument can be made for the off-diagonal matrix elements. Their width Γ is reduced by a factor of $\exp(-2|n - m|/\xi_1)$. As shown in Fig. 2, these conclusions are well supported by numerical results.

We argue that, for sufficiently large u, the resulting banded-random-matrix model predicts $\xi_2 \sim \xi_1^2$ in agreement with Eq. (1) when neglecting correlations between the matrix elements. This result would follow immediately from analytical results for banded random matrices if the distribution of the matrix elements had a finite variance. In this case, the localization length is proportional to the square of the bandwidth [13]. The same result holds true for banded Cauchy matrices for the following reason: Since the eigenstates of the banded Cauchy matrix are localized, they effectively sample only a finite number

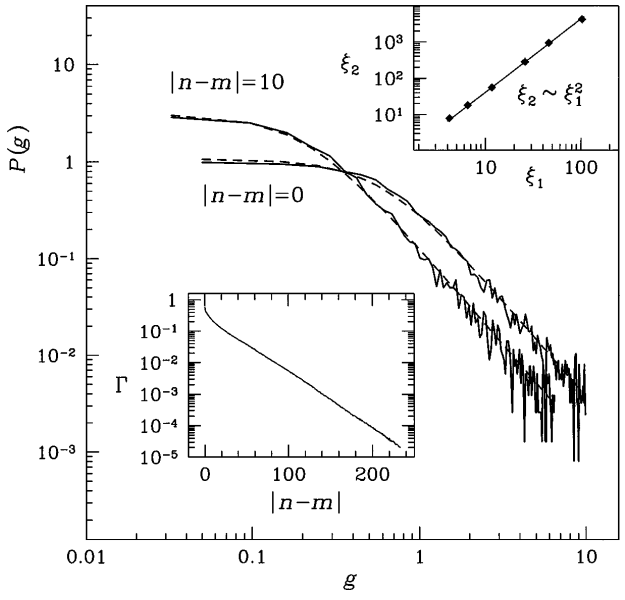

FIG. 2. Distributions of diagonal and off-diagonal matrix elements g of the projected two-particle Green function $\tilde{G}_0$ for $\xi_1 = 46.6$ (solid lines). The dashed lines are fits by Cauchy distributions. Their width Γ decreases exponentially away from the diagonal on the scale $\xi_1/2$ as shown in the lower inset. The upper inset shows ξ_2 vs ξ_1 as obtained from the banded-random-matrix model discussed in the text. These data obtained for systems with 10^6 sites confirm our arguments that $\xi_2 \sim \xi_1^2$ in this model.

of matrix elements drawn from the Cauchy distribution. Hence, there exists a corresponding typical largest matrix element $g_{\max}$ [12]. Beyond $g_{\max}$ the Cauchy distribution can be cut off, and the resulting effective distribution of matrix elements has a finite variance. This implies that banded Cauchy matrices belong to the same universality class as ordinary banded random matrices. We confirmed this conclusion numerically by computing the localization length of banded Cauchy matrices as a function of bandwidth as shown in the upper inset of Fig. 2. We have also studied the u dependence of ξ_2 predicted by this banded-random-matrix model. However, we find that the ξ_2 computed from this model does *not* exhibit the scaling (9) found for the exact solution. Presumably, this is due to correlations between the matrix elements of $\tilde{G}_0$. For example, correlations in the exact $\tilde{G}_0$ are implied by Eq. (8) for exceptionally large matrix elements. Large matrix elements are due to small energy denominators. Each product state with energy close to E leads to a correlated $\xi_1 \times \xi_1$ block of large matrix elements in $\tilde{G}_0$. Such correlations are neglected in the banded-random-matrix model.

It is an interesting problem to study two-particle localization in more than one dimension. In the absence of an understanding of the physical origin of the scaling parameter u/Δ_ξ, it is not clear how to generalize our results to these cases. For quasi-one-dimensional wires with a finite number of channels M, one easily derives a generalized banded-random-matrix-model. Assuming that this banded-random-matrix model again correctly predicts the dependence of ξ_2 on the bandwidth and combining the result with scaling suggests that the scaling function remains linear. For quasi-one-dimensional wires we can order the doubly occupied sites sequentially along the longitudinal direction. When ξ_1 is larger than the transverse dimensions of the wire, the bandwidth of $\tilde{G}_0$ is equal to $M\xi_1$, yielding a corresponding "localization length" $(M\xi_1)^2$. The actual localization length in the longitudinal direction is smaller by a factor M, hence $\xi_2 \sim M\xi_1^2$. Finally, assuming the above scaling behavior and noting that $\Delta_\xi \sim t/M\xi_1$, we obtain a linear scaling function, $\xi_2/\xi_1 \sim |u|/\Delta_\xi$.

In summary, we have studied the interaction-induced delocalization of two particles in a one-dimensional random potential by a novel and efficient numerical approach. We have found that the two-particle localization length ξ_2 for coherent propagation of the two particles satisfies the scaling relation $\xi_2/\xi_1 = f(u/\Delta_\xi)$ as a function of interaction strength u and one-particle localization length ξ_1. This implies that the effect sets in for weaker interactions than previously predicted. At the center of the band our data suggest that the scaling function is linear. At present, we do not have a good physical understanding of this unexpected scaling behavior. It will be interesting to see whether the scaling found in this paper can be generalized to higher dimensions or whether it is a specific feature of one dimension.

It would also be interesting to study implications of coherent propagation due to interactions at finite particle density. Our numerical approach can be extended to study the propagation of quasiparticle pairs in the Anderson insulator. This will be the subject of a separate publication.

One of us (F. v. O.) is particularly indebted to B. Altshuler and A. Luther for important discussions which eventually led to a significant revision of a previous version of the manuscript. We also acknowledge useful discussions with G. Hackenbroich, B. Huckestein, A. Müller-Groeling, D. L. Shepelyansky, and H. A. Weidenmüller.

[1] For a review, see B. Kramer and A. MacKinnon, Rep. Prog. Phys. **56**, 1469 (1993).
[2] For a review, see D. Belitz and T. R. Kirkpatrick, Rev. Mod. Phys. **66**, 261 (1994).
[3] O. N. Dorokhov, Zh. Eksp. Teor. Fiz. **98**, 646 (1990) [Sov. Phys. JETP **71**, 360 (1990)].
[4] D. L. Shepelyansky, Phys. Rev. Lett. **73**, 2607 (1994); F. Borgonovi and D. L. Shepelyansky, Nonlinearity **8**, 877 (1995).
[5] Y. Imry, Europhys. Lett. **30**, 405 (1995).
[6] K. Frahm, A. Müller-Groeling, J.-L. Pichard, and D. Weinmann, Europhys. Lett. **31**, 169 (1995); D. Weinmann, A. Müller-Groeling, J.-L. Pichard, and K. Frahm, Phys. Rev. Lett. **75**, 1598 (1995).
[7] B. Huckestein, Rev. Mod. Phys. **67**, 357 (1995).
[8] Note that this Green function is properly symmetrized although this may not be apparent because of the projection onto doubly occupied sites.
[9] We have not included our data for $W = 1$ ($\xi_1 = 105$) in Fig. 1. While they are consistent with a linear scaling function f, the corresponding error bars are rather large because ξ_2 is comparable to the system size.
[10] F. von Oppen and T. Wettig (unpublished).
[11] Note that the dependence of ξ_2 on ξ_1 involves both a linear and a quadratic term. From a double-logarithmic plot of ξ_2 vs ξ_1 over a limited parameter range, this dependence was identified in Ref. [6] as a power law with noninteger exponent between one and two.
[12] E.g., see J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[13] Y. V. Fyodorov and A. D. Mirlin, Phys. Rev. Lett. **67**, 2405 (1991).