Analytic Evaluation of the Orthopositronium-to-Three-Photon Decay Amplitudes to One-Loop Order

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The three independent amplitudes describing the decay of orthopositronium to three photons are evaluated in analytic form to one-loop order. The amplitudes are used to obtain the order-α and order-α^2 "square" contributions to the orthopositronium decay rate as integrals over the two-dimensional phase space. The results for the decay rate contributions are $\Gamma_1 = -10.286\,606(10)\,(\alpha/\pi)\Gamma_{\rm LO}$ and $\Gamma_2(\text{square}) = 28.860(2)\,(\alpha/\pi)^2\Gamma_{\rm LO}$, where $\Gamma_{\rm LO}$ is the lowest order rate. [S0031-9007(96)00486-3]

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The calculation of order-α^2 corrections to the orthopositronium (o-Ps) decay rate is one of the most important outstanding problems in bound-state QED. The experimental situation is not clear. The two highest precision measurements [1,2] disagree strongly with the order-α theoretical prediction, and seem to imply an unusually large value for the still unknown order-α^2 coefficient. A more recent measurement [3] is in agreement with the order-α result.

In this Letter, I report on a new approach to the calculation of the o-Ps decay rate. I use a formalism of covariant decay amplitudes developed in the study of Z boson decay to three photons. I have been able to evaluate the one-loop o-Ps $\to 3\gamma$ amplitudes analytically in terms of dilogarithms and other elementary functions. Using these one-loop amplitudes, I obtained greatly improved values for the order-α correction to the decay rate and the part of the order-α^2 correction to the decay rate coming from the squares of the order-α amplitudes. Knowledge of the one-loop amplitudes allows one to find an analytic form for the one-loop differential decay distribution, and a convenient means for accurately calculating the one-loop photon energy spectrum.

In most prior work on the o-Ps decay rate, the square of the decay matrix element was computed directly [4–8]. An alternative amplitude based method was used by Burichenko in his earlier calculation of the "square" contribution to the order-α^2 rate [9].

The decay of the massive vector particle orthopositronium to three photons is analogous to the decay of the Z boson to three photons. A convenient formalism for the $Z \to 3\gamma$ process was given by Glover and Morgan [10]. They used Bose symmetry and gauge invariance to show that the matrix element

$$M = \epsilon^*_{1\mu_1}\epsilon^*_{2\mu_2}\epsilon^*_{3\mu_3}\epsilon_\alpha M^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3) \tag{1}$$

for the decay of a vector particle (momentum P, polarization ϵ) to three photons (momenta k_i, polarizations ϵ_i) can be written in terms of just three independent amplitudes. After writing the decay tensor in the manifestly symmetric form

$$M^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3) = \sum_{S_3} \mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3)\,, \tag{2}$$

where the sum is over the six permutations of the photon momentum vectors, they showed that $\mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3)$ can be written in terms of amplitudes A_1, A_2, A_3 as

$$\begin{aligned}\mathcal{M}^{\mu_1\mu_2\mu_3\alpha}(k_1,k_2,k_3) = {} & A_1(k_1,k_2,k_3)\,\frac{1}{k_1\cdot k_3}\left(\frac{k_3^{\mu_1}k_1^{\mu_3}}{k_1\cdot k_3} - g^{\mu_1\mu_3}\right)k_1^\alpha\left(\frac{k_3^{\mu_2}}{k_2\cdot k_3} - \frac{k_1^{\mu_2}}{k_1\cdot k_2}\right) + A_2(k_1,k_2,k_3) \\ & \times\left\{\frac{1}{k_2\cdot k_3}\left(\frac{k_1^\alpha k_3^{\mu_1}}{k_1\cdot k_3} - g^{\alpha\mu_1}\right)\left(\frac{k_1^{\mu_2}k_2^{\mu_3}}{k_1\cdot k_2} - g^{\mu_2\mu_3}\right) + \frac{1}{k_1\cdot k_3}\left(\frac{k_1^{\mu_2}}{k_1\cdot k_2} - \frac{k_3^{\mu_2}}{k_2\cdot k_3}\right)\right. \\ & \left.\times\,(k_1^{\mu_3}g^{\alpha\mu_1} - k_1^\alpha g^{\mu_1\mu_3})\right\} + A_3(k_1,k_2,k_3)\,\frac{1}{k_1\cdot k_3}\left(\frac{k_1^\alpha k_3^{\mu_1}}{k_1\cdot k_3} - g^{\alpha\mu_1}\right)\left(\frac{k_3^{\mu_2}k_2^{\mu_3}}{k_2\cdot k_3} - g^{\mu_2\mu_3}\right).\end{aligned} \tag{3}$$

The o-Ps $\to 3\gamma$ decay rate is an integral over the two-dimensional phase space [11]

$$\Gamma = \frac{m}{768\pi^3}\int_0^1 dx_1\int_{1-x_1}^1 dx_3 \sum_{\lambda_1,\lambda_2,\lambda_3}\frac{1}{3}\sum_m |M_{\lambda_1,\lambda_2,\lambda_3;m}|^2, \tag{4}$$

where $x_i = E_i/W$ is the normalized energy of the ith photon [with $W = m + O(m\alpha^2)$ equal to half of the o-Ps mass] in the o-Ps rest frame, and $M_{\lambda_1,\lambda_2,\lambda_3;m}$ is the matrix element for o-Ps with spin component m to decay to three photons with helicities $\lambda_1, \lambda_2, \lambda_3$. Glover and Morgan showed how to calculate $M_{\lambda_1,\lambda_2,\lambda_3;m}$ in terms of the amplitudes A_1, A_2, A_3 in a two-photon center-of-mass frame by using explicit forms for the photon and o-Ps spin vectors. Since the spin averaged and helicity summed square of the matrix element is an invariant, it can be used in the o-Ps rest frame as well.

The decay amplitudes can be written as

$$A_i = A_i^{(0)} + A_i^{(1)} + A_i^{(2)} + \cdots \tag{5}$$

for $i = 1, 2, 3$, where the superscript indicates the power of α above that of the lowest-order amplitudes $A_i^{(0)} = A_i^{\mathrm{LO}}$. [Terms of order $A_i^{(2)}$ and higher also contain factors of $\ln(1/\alpha)$.] The expressions for the squares $|M_{\lambda_1,\lambda_2,\lambda_3;m}|^2$ contain parts of the form

$$\begin{aligned} A_i^* A_j = {} & A_i^{(0)*} A_j^{(0)} + [A_i^{(0)*} A_j^{(1)} + A_i^{(1)*} A_j^{(0)}] + A_i^{(1)*} A_j^{(1)} \\ & + [A_i^{(0)*} A_j^{(2)} + A_i^{(2)*} A_j^{(0)}] + \cdots \end{aligned} \tag{6}$$

for various combinations of i and j. The $A_i^{(0)*} A_j^{(0)}$ terms give the lowest-order differential decay distribution. The $A_i^{(0)*} A_j^{(1)} + A_i^{(1)*} A_j^{(0)}$ terms give the order-α correction, and the $A_i^{(1)*} A_j^{(1)}$ and $A_i^{(0)*} A_j^{(2)} + A_i^{(2)*} A_j^{(0)}$ terms give the order-α^2 corrections.

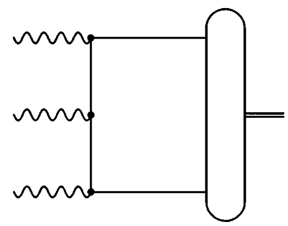

FIG. 1. The lowest-order o-Ps $\to 3\gamma$ decay graph.

The lowest-order o-Ps $\to 3\gamma$ amplitudes in this formalism are particularly simple. The lowest-order decay matrix element in the o-Ps rest frame, from the diagram of Fig. 1, is [7,8]

$$\begin{aligned} M_{\mathrm{LO}} = {} & -\frac{i\pi\alpha^3}{m^2} \sum_{S_3} \frac{1}{x_1 x_3} \frac{1}{4} \\ & \times \mathrm{tr}\{\gamma\epsilon_3[\gamma(-p + k_3) + m]\gamma\epsilon_2[\gamma(p - k_1) + m] \\ & \times \gamma\epsilon_1(\gamma N + 1)\gamma\epsilon(\gamma N - 1)\}, \end{aligned} \tag{7}$$

where $p = mN$ with $N = (1, \vec{0})$ the timelike unit vector. The amplitudes A_1, A_2, A_3 are found by calculating the trace for M_{LO} and identifying the appropriate parts. One writes the tensor matrix element as

$$\begin{aligned} M^{\mu_1\mu_2\mu_3\alpha}(k_1, k_2, k_3) = {} & -A_1(k_1, k_2, k_3) k_3^{\mu_1} k_1^{\mu_2} k_1^{\mu_3} k_1^{\alpha} [(k_1 \cdot k_3)^2 (k_1 \cdot k_2)]^{-1} + A_2(k_1, k_2, k_3) k_3^{\mu_1} k_1^{\mu_2} k_2^{\mu_3} k_1^{\alpha} \\ & \times [(k_1 \cdot k_2)(k_2 \cdot k_3)(k_3 \cdot k_1)]^{-1} + A_3(k_1, k_2, k_3) k_3^{\mu_1} k_3^{\mu_2} k_2^{\mu_3} k_1^{\alpha} [(k_1 \cdot k_3)^2 (k_2 \cdot k_3)]^{-1} + \cdots \end{aligned} \tag{8}$$

and finds A_1 by taking the coefficient of $k_3^{\mu_1} k_1^{\mu_2} k_1^{\mu_3} k_1^{\alpha}$, etc. The result is

$$A_1^{\mathrm{LO}}(x_1, x_2, x_3) = 0, \tag{9a}$$

$$A_2^{\mathrm{LO}}(x_1, x_2, x_3) = 16 i\pi\alpha^3 \frac{\bar{x}_1 \bar{x}_2 \bar{x}_3}{x_1 x_2 x_3}, \tag{9b}$$

$$A_3^{\mathrm{LO}}(x_1, x_2, x_3) = 0, \tag{9c}$$

in terms of the variables x_1, x_2, x_3 with $\bar{x}_k = 1 - x_k$. These variables satisfy the energy conservation law $x_1 + x_2 + x_3 = 2$, and give the scalar products $k_i \cdot k_j = 2\bar{x}_k$ with $\{i, j, k\}$ any permutation of $\{1, 2, 3\}$. The integral of Eq. (4) for Γ_{LO} can be done analytically, giving the Ore and Powell result for the lowest-order rate [12]

$$\Gamma_{\mathrm{LO}} = \frac{2}{9\pi} (\pi^2 - 9) m\alpha^6. \tag{10}$$

The graphs contributing to the order-α corrected decay amplitudes in the renormalized Coulomb gauge are shown in Fig. 2. It is most convenient to actually calculate these graphs with Feynman gauge virtual photons. As a first step in the transformation from Coulomb gauge to Feynman gauge virtual photons, I add back in the Coulomb gauge wave function renormalization counterterm $B_C^{(1)} = -(\alpha/4\pi)(4\pi\mu^2/m^2)^\epsilon \Gamma(\epsilon)$ [13], where the wave function renormalization constant is $Z_{2C} = 1 + B_C^{(1)} + O(\alpha^2)$. This only affects the self-energy, inner vertex, and outer vertex graphs, making them unrenormalized graphs. There is a net $B_C^{(1)} M_{\mathrm{LO}}$ left over. Next, I replace Coulomb gauge photons by Feynman gauge photons through the use of

$$D_{\mu\nu}^C(\ell) = D_{\mu\nu}^F(\ell) + b_\mu(\ell)\ell_\nu + \ell_\mu b_\nu(\ell), \tag{11a}$$

$$b_\mu(\ell) = \frac{1}{2\ell^2 \vec{\ell}^2} [-\ell_\mu + 2(\ell \cdot N) N_\mu]. \tag{11b}$$

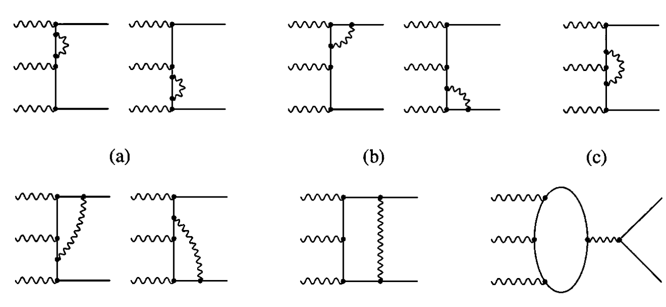

FIG. 2. Graphs contributing to the o-Ps decay amplitudes through order-α. They are the (a) self-energy, (b) outer vertex, (c) inner vertex, (d) double vertex, (e) ladder, and (f) annihilation contributions. The ladder graph (e) contains the lowest order amplitudes as well as order-α corrections. The wave function factors are implicit in these graphs.

The lowest-order Ward identity

$$\gamma\ell = [\gamma(q + \ell) - m] - (\gamma q - m) \tag{12}$$

is used to combine the longitudinal terms into a single "gauge correction term," which is [7]

$$M_{GC} = \frac{\alpha}{\pi}(-2\ln\alpha - 1)M_{\mathrm{LO}}\,. \tag{13}$$

The virtual photons in the graphs of Fig. 2 are now in Feynman gauge.

The ladder graph Fig. 2(e) requires special care in its evaluation since it contains the lowest order contribution. This contribution can be identified and subtracted out, as discussed in detail in Refs. [7,8]. The result is that

$$M_L = \left\{1 + \frac{\alpha}{\pi}(2\ln\alpha - 1)\right\}M_{\mathrm{LO}} + M_{LS}. \tag{14}$$

The subtracted ladder graph is

$$\begin{aligned} M_{LS} = -i\alpha^4 m^2 \sum_{S_3}\int \frac{d^4\ell}{i\pi^2} \\ \times [\ell^2(\ell^2 - 2\ell p)(\ell^2 + 2\ell p)Z(\ell)]^{-1} \\ \times \left\{[\mathrm{tr}(\ell) - \mathrm{tr}(0)] - \frac{\mathrm{tr}(0)}{Z(0)}[Z(\ell) - Z(0)]\right\}, \end{aligned} \tag{15}$$

with

$$\begin{aligned} \mathrm{tr}(\ell) = \tfrac{1}{4}\,\mathrm{tr}\{\gamma^\mu[\gamma(\ell - p) + m]\gamma\epsilon_3[\gamma(\ell - p + k_3) + m]\gamma\epsilon_2[\gamma(\ell + p - k_1) + m]\gamma\epsilon_1[\gamma(\ell + p) + m] \\ \times \gamma_\mu(\gamma N + 1)\gamma\epsilon(\gamma N - 1)\}, \end{aligned} \tag{16}$$

and

$$Z(\ell) = [(\ell - p + k_3)^2 - m^2][(\ell + p - k_1)^2 - m^2]. \tag{17}$$

The subtraction in Eq. (15) takes away the ℓ-independent part of $\mathrm{tr}(\ell)/Z(\ell)$, which has an infrared singularity. This binding singularity, appropriately regulated by the binding energy and nonvanishing relative three-momentum of the bound state, produced the lowest-order contribution to M_L shown in Eq. (14).

I used the Passarino-Veltman formalism [14] to systematize the evaluation of the one-loop momentum space integrals. In this approach, tensor integrals like

$$\int \frac{d^4\ell}{i\pi^2}\,\frac{\ell^\mu\ell^\nu\cdots}{[-\ell^2 + m_1^2][-(\ell + p_1)^2 + m_2^2]\cdots} \tag{18}$$

are reduced algebraically to expressions containing only scalar integrals of the form

$$\int \frac{d^4\ell}{i\pi^2}\,\frac{1}{[-\ell^2 + m_1^2][-(\ell + p_1)^2 + m_2^2]\cdots}\,. \tag{19}$$

The scalar integrals are the only ones that must actually be integrated. A slight modification of this procedure is required for the ladder graph since the corresponding scalar integral contains a binding singularity. There, the procedure was based on the vector integral

$$\begin{aligned} \int \frac{d^4\ell}{i\pi^2}\, m^6\ell^\mu\{[-\ell^2][-\ell^2 - 2\ell p][-\ell^2 + 2\ell p] \\ \times [-(\ell + p - k_1)^2 + m^2][-(\ell - p + k_3)^2 + m^2]\}^{-1} \\ = G_{11}p^\mu + G_{12}k_1^\mu + G_{13}k_3^\mu. \end{aligned} \tag{20}$$

Performing this vector integral for the G_{1i} functions was the heart of the calculation. I found that

$$\begin{aligned} G_{11}(x_1,x_3) = \frac{1}{8\bar{x}_1}[I_0(x_1,x_3) + I_1(x_1,x_3)] \\ - \frac{1}{8\bar{x}_3}[I_0(x_3,x_1) + I_1(x_3,x_1)], \end{aligned} \tag{21a}$$

$$\begin{aligned} G_{12}(x_1,x_3) = \frac{1}{16x_1\bar{x}_1}[(1 - 2x_1)I_0(x_1,x_3) - I_1(x_1,x_3)] \\ + \frac{1}{16x_1\bar{x}_3}[I_0(x_3,x_1) + I_1(x_3,x_1)], \end{aligned} \tag{21b}$$

$$\begin{aligned} G_{13}(x_1,x_3) = -\frac{1}{16\bar{x}_1x_3}[I_0(x_1,x_3) + I_1(x_1,x_3)] \\ - \frac{1}{16x_3\bar{x}_3}[(1 - 2x_3)I_0(x_3,x_1) - I_1(x_3,x_1)], \end{aligned} \tag{21c}$$

where

$$I_0(x_1,x_3) = \frac{1}{\sqrt{x_1\bar{x}_1x_3\bar{x}_3}}[\mathrm{Li}_2(r_+,\theta) - \mathrm{Li}_2(r_-,\theta)], \tag{22a}$$

$$I_1(x_1,x_3) = \frac{1}{(x_1 - x_3)}\ln\left(\frac{x_1}{x_3}\right) - \frac{2}{\sqrt{x_3\bar{x}_3}}\arctan\left(\sqrt{\frac{\bar{x}_3}{x_3}}\right), \tag{22b}$$

with $r_\pm = \sqrt{\bar{x}_1} \pm \sqrt{x_1\bar{x}_3/x_3}$ and $\theta = \arctan\sqrt{x_1/\bar{x}_1}$. The dilogarithm function $\mathrm{Li}_2(r,\theta)$ is discussed in Lewin [15]. Difficulty in the numerical evaluation of this vector integral was the main cause of uncertainty in previous calculations of the one-loop corrections to the decay rate [5–8]. With the ladder graph vector functions and the remaining scalar functions in hand, the rest of the one-loop integrals were found algebraically using a routine written in the MATHEMATICA programming language.

The amplitudes for the six order-α decay graphs were evaluated one by one, then summed along with the contributions from the renormalization term $B_C^{(1)} M_{\mathrm{LO}}$ and the gauge correction term (13). The resulting expressions are sums of rational functions of the x_i times logarithms, dilogs, and inverse tangent functions. Of course, the ultraviolet divergence cancels in the sum. The expressions for the amplitudes are moderately long, and will be given in another place.

Practical results follow immediately. The two-dimensional integral for the order-α correction to the o-Ps $\rightarrow 3\gamma$ decay rate gives [16]

$$\Gamma_1 = -10.286\,606(10)\,\frac{\alpha}{\pi}\,\Gamma_{\mathrm{LO}}\,. \tag{23}$$

This represents a 60-fold improvement in precision over the previous best result, which had a coefficient of $-10.2866(6)$ [8]. The two-dimensional integral for the part of the order-α^2 correction to the decay rate coming from the $A_i^{(1)*} A_j^{(1)}$ terms gives [17]

$$\Gamma_2(\mathrm{square}) = 28.860(2)\left(\frac{\alpha}{\pi}\right)^2 \Gamma_{\mathrm{LO}}\,. \tag{24}$$

The previous result for this contribution was 28.8(2) [9].

The amplitude based approach is algebraically more efficient than the method of calculating the square of the matrix element directly, and should see useful service in the evaluation of the remaining order-α^2 contributions to the o-Ps decay rate.

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[17] Quadruple precision was required in order to retain numerical significance in some parts of the Γ_2(square) integrand. The reported result was obtained with approximately 10^5 function calls.