Diffusion of Periodically Forced Brownian Particles Moving in Space-Periodic Potentials

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The diffusion properties of overdamped particles moving in spatially symmetrical periodic potentials subject to both symmetrical time-periodic driving and stochastic forcing are investigated [the typical model formally reads $\dot{x} = -V'(x) + U(t) + \Gamma(t)$, $V(L \pm x) = V(x)$, $U(T + t) = U(t)$, $\langle \Gamma(t) \rangle = 0$, $\langle \Gamma(t)\Gamma(\tau) \rangle = 2D\delta(t - \tau)$]. It is found that the diffusion rate can be greatly enhanced if the various forcings are chosen in an optimal matching. In particular, we may get a diffusion rate larger than the rate of free diffusion. [S0031-9007(96)00420-6]

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Recently, there has been great interest in studying influences of symmetrical forces on transport properties. In particular, the works on the forced ratchets showed [1–4] that if a particle moves in an asymmetrical periodic potential and is forced by a symmetrical thermal noise and an also symmetrical force (which may be stochastic or deterministic) with long enough correlation time, a net current in a certain direction (according to the asymmetry of the potential) may occur in absence of an appropriate bias or thermal gradients. In this case, the time-correlated force (the force breaks the equilibrium condition) plays an important role in essentially changing the properties of thermal systems. For the forced ratchets the asymmetry of the potential is, of course, the key point for the net macroscopic current. How do "nonequilibrium" symmetrical correlated forces influence thermal systems when potentials are symmetric? Are there some new features appearing in these symmetrical potential cases which are of fundamental significance for the relaxation processes? In this Letter, we focus on Brownian particles moving in symmetrical periodic potentials, subject to time-periodic forcings and thermal noise. Specifically, we investigate the system

$$\begin{aligned} &\dot{x} = -V'(x) + U(t) + \Gamma(t), \\ &V(L \pm x) = V(x), \quad U(T + t) = U(t), \\ &\langle \Gamma(t) \rangle = 0, \quad \langle \Gamma(t)\Gamma(\tau) \rangle = 2D\delta(t - \tau), \end{aligned} \tag{1}$$

where $V'(x)$ denotes the derivative of $V(x)$ with respect to x. We are mainly interested in the influence of combined actions of the space-periodic potential $V(x)$ and the time-periodic field $U(t)$ on the diffusion behavior of the system.

A fundamental and well understood problem in physics is the diffusion of free Brownian motion

$$\dot{x} = \Gamma(t). \tag{2}$$

Its exact solution in terms of probability distributions with an initial state that is given by $\rho(x,0) = \delta(x - x_0)$ can immediately be written as $\rho(x,t) = (1/\sqrt{2Dt}) \exp\{-(x - x_0)^2/2Dt\}$. Hence, the diffusion rate reads

$$\langle \Delta x(t)^2 \rangle - \langle \Delta x(0)^2 \rangle = 2Dt, \tag{3}$$

with $\langle \Delta x^2 \rangle := \langle x^2 \rangle - \langle x \rangle^2$. In practice, one often wants or even needs to control the diffusion rate with given fixed noise level (e.g., fixed temperature) by applying certain deterministic forces. Obviously, the diffusion rate will be reduced by introducing various attracting forces. The problem of whether one can enhance the diffusion rate exceeding the diffusion rate of free Brownian motion by applying finite deterministic drivings has attracted only a little attention so far. However, this problem is of great theoretical significance and practical importance. The most natural manipulation in this regard is to incorporate certain time-dependent or space-dependent forces.

Indeed, a time-dependent forcing $[\dot{x} = U(t) + \Gamma(t)]$ alone will not suffice to produce the diffusion relation (3). As already mentioned with a space-periodic potential only $[\dot{x} = -V'(x) + \Gamma(t)]$ the diffusion process will be obstructed by potential barriers, and the diffusion rate can be simply reduced with respect to (3). A further test is to combine both the time-periodic force and the space-periodic gradient. We note that diffusion can be greatly accelerated if we incorporate the two forces by a proper choice of the control parameters.

In order to show this we start from Eq. (1). For the sake of simplicity and without any loss of generality we consider a sinusoidal potential and a square wave periodic force

$$\begin{aligned} &V(x) := -\cos(x), \\ &U(t) := \begin{cases} A \text{ for} & qT < t \le qT + \frac{T}{2}, \\ -A \text{ for} & qT + \frac{T}{2} < t \le (q + 1)T. \end{cases} \end{aligned} \tag{4}$$

Even for the simple form (4) we cannot obtain an exact expression for the diffusion rate. This would require the explicit time-dependent solution of (1) which can only be approximated for small D. On this account the Langevin equation (LE) (1) is transformed into a Fokker-Planck

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equation (FPE)

$$\frac{\partial}{\partial t}\rho(x,t) = \frac{\partial}{\partial x}[\sin(x) - U(t)]\rho(x,t) + D\frac{\partial^2}{\partial x^2}\rho(x,t), \tag{5}$$

and for $D \ll 1$, $T \gg 1$ we can now reduce this FPE to a master equation (ME)

$$\dot{P}_i = \begin{cases} -[R_+(r) + R_+(l)]P_i + R_+(r)P_{i-1} + R_+(l)P_{i+1}, & qT < t \le qT + \frac{T}{2}, \\ -[R_-(r) + R_-(l)]P_i + R_-(r)P_{i-1} + R_-(l)P_{i+1}, & qT + \frac{T}{2} < t \le (q+1)T, \end{cases} \tag{6}$$

where $i = \ldots, -2, -1, 0, 1, 2, \ldots$. P_i is the total probability contained in the ith potential well, given by $P_i(t) := \int_{\pi+2(i-1)\pi}^{\pi+2i\pi} \rho(x,t)\,dx$. Besides, the transition rates $R_\pm(r,l)$ read

$$R_\pm(r) = R_\mp(l) = \frac{1}{2\pi}\exp\left\{-\frac{2 \mp \pi A}{D}\right\}. \tag{7}$$

Inserting $f(s,t) := \sum_{n=-\infty}^{\infty} s^n P_n(t)$ into Eq. (6) we obtain its exact time-dependent solution for $qT < t \le qT + \frac{T}{2}$ as

$$f(s,t) = f(s,0)[F^+(s)F^-(s)]^q \exp\{[R_+(r)s + R_+(l)/s - R_+(r) - R_+(l)](t - qT)\}, \tag{8a}$$

and in complete analogy, for $qT + \frac{T}{2} < t \le (q+1)T$,

$$f(s,t) = f(s,0)F^+(s)^{q+1}F^-(s)^q \exp\{[R_-(r)s + R_-(l)/s - R_-(r) - R_-(l)](t - qT - \tfrac{T}{2})\}, \tag{8b}$$

with

$$F^\pm(s) := \exp\{(T/2)[R_\pm(r)s + R_\pm(l)/s - R_\pm(r) - R_\pm(l)]\}.$$

The fluctuation follows from Eqs. (8a) and (8b) as

$$\begin{aligned}\langle\Delta n^2\rangle_t - \langle\Delta n^2\rangle_{t=0} &= [sf'(s,t)]'_{s=1} - f'(s,t)^2_{s=1} \\ &\quad - \{[sf'(s,0)]'_{s=1} - f'(s,0)^2_{s=1}\} \\ &= R(A)t, \\ \text{with } R(A) &:= R_+(r) + R_+(l) \\ &= R_-(r) + R_-(l), \end{aligned} \tag{9}$$

where the notation prime here represents the derivative with respect to s. Note that Eq. (9) has exactly the same form as (3) with the diffusion rate changed from $2D$ to $2R(A)$. With $R(A)/R(A=0) = \cosh(\pi A/D) > 1$, we can conclude that periodic modulation is preferable for an enhancement of the diffusion rate for the case of $D \ll 1$.

The given analysis is exact for the ME (6), while it only approximates the FPE (5) under the conditions $D \ll 2 \pm \pi A$ and $T \gg 1$. In order to study the diffusion rate in more general cases, especially looking towards $R(A) > D$, we have to invoke numerical simulations. In order to integrate (1) numerically we produce white noise $\Gamma(t)$ with the Box-Müller formula [5]. Note that the integration is realized by a single step Euler method with time step Δt so that we have to multiply $\Gamma(t)$ by $\Delta t^{(1/2)}$ [6]. Thus, in Figs. 1 and 2, we simulate Eq. (1) with $V(x)$ and $U(t)$ given by (4). We plot $\langle\Delta x^2\rangle$ vs t for various D and various A, respectively. Each data point is obtained by averaging over 10^4 integrations. The linear dependence of $\langle\Delta x^2\rangle$ on t is obvious, and shows typical diffusion processes. In addition, we measure the quantity $\eta := R(A)/D$ to describe the influences of A and D on the diffusion rate in a more quantitative manner.

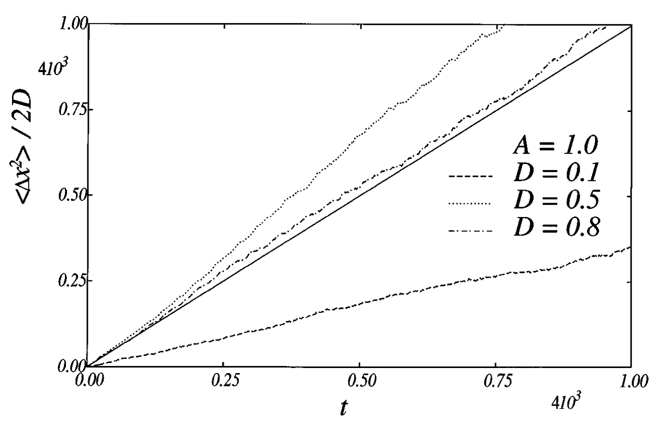

FIG. 1. Simulations of Eqs. (1) and (4) with $T = 10$ (the same T will be used for this figure, as well as for Figs. 2–4), $A = 1.0$. $\langle\Delta x^2\rangle$ is plotted vs t for various D's.

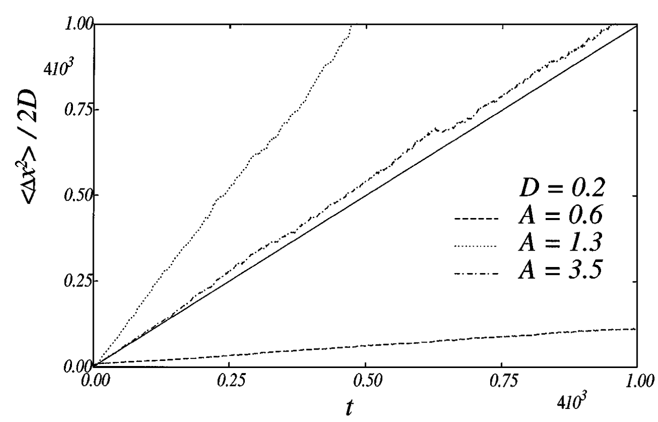

FIG. 2. $D = 0.2$. $\langle\Delta x^2\rangle$ is plotted vs t for various A's. In both Fig. 1 and this figure standard linear diffusion behavior is observed.

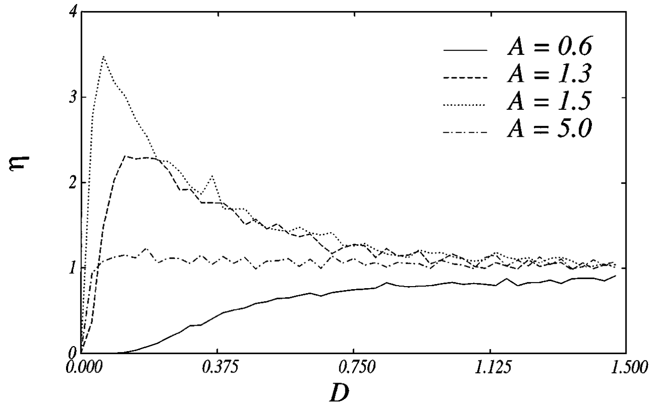

FIG. 3. $\eta := R(A)/D$ is plotted vs D for various A's.

In Figs. 3,4 we plot η against D, A for various A, D. Discussing these results in more detail, at first, we find $\eta \rightarrow 1$ for the case $D \gg 1$, or $A \gg 1$. This result belongs to the fact that the system does not realize the existence of the space-periodic potential for large D or A, i.e., that the diffusion rate should reach Eq. (3). Moreover, for $D \ll 1$ the quantity η increases as A increases in the vicinity of small A—note the good accordance with the theoretical result (9). It is remarkable that some curves are peaking at certain parameter values. This manifests the existence of optimal D's (Fig. 3), and A's (Fig. 4) for the enhancement of the diffusion rate. Certainly, these phenomena recall the effect of stochastic resonance (SR) [7–16], whereby now this "SR" refers to the acceleration of diffusion. That means a new diffusion mechanism with combined actions of noise, and certain properly chosen finite deterministic forces can be much more effective than that of free Brownian motion, since—as the most important result—η may exceed unity in a large region around some optimal parameter regions. There is an intuitive understanding of this higher efficiency: The optimal matching of the periodic force and noise may drive the probability peaks up to the potential hills during each

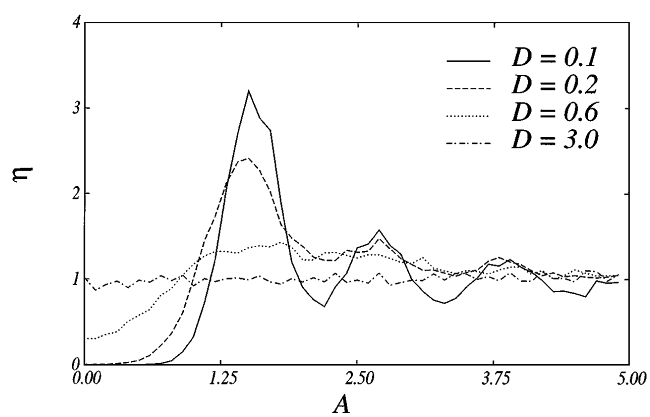

FIG. 4. η vs A for various D's. In both Fig. 3 and this figure one finds peaked response curves of the diffusion rate, and the diffusion rate can be much larger than that of free diffusion at optimal matching of A and D.

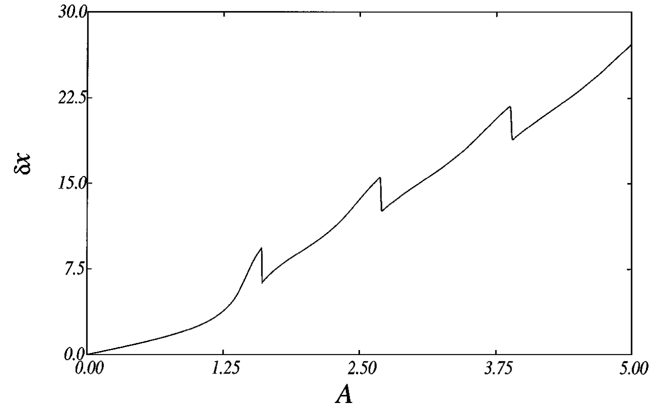

FIG. 5. $\delta x := x_{\max} - x_{\min}$ is plotted vs A for $D = 0$. The high slope segments coincide with the peaks in Fig. 4.

time period. These peaks scatter at the potential barriers and are smashed into small pieces during the scattering. Finally, the probability diffuses very quickly into wide regions. For rather small A and D, however, it is difficult to push the probability up to potential hills, whereas for too large A and D the Brownian particle simply passes over the hills. In both latter cases the scattering with potential barriers is very weak, and the probabilities are kept more centralized. Then the diffusion rates should remain small. Without the spatial structure, such a scattering does not occur at all. Therefore, all three forces, spatially periodic gradients, time-periodic modulations, and stochastic stirrings are necessary, and only their optimal collective actions generate the above features. Likewise, it is interesting to see that the $\eta - A$ curves are multipeaked for small D. The first and highest peak corresponds to the situation that the probability may be driven from a potential basin to the nearest potential hill in a half period. The second, third, etc. peaks correspond to the situation that the probability peak may be driven to the second, third, etc. neighbor potential hills, respectively, in a half period (like higher harmonics).

In Fig. 5 we plot $\delta x := x_{\max} - x_{\min}$ vs A for $D = 0$ and $T = 10$, where $x_{\max}$ and $x_{\min}$ are the maximum and

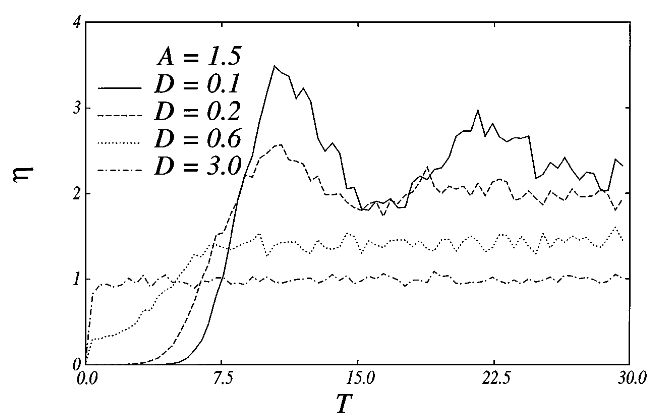

FIG. 6. η is plotted vs T for $A = 1.5$ and various D's. The response curves are multipeaked; compare Fig. 4.

minimum x's in the asymptotic state. In the figure the high slope segments correspond to an increase of the number of basins the system visits. These segments coincide with the peaks of the response curves in Fig. 4. That shows the close relationship between the anomalous enhancement of the diffusion rate and the relaxation processes around potential barriers.

In Fig. 6 we plot η against T for fixed A and various D. Again we find multipeak behavior, in particular, for small D's. The reason here is the same as the multipeak structure in Fig. 4.

Finally, we would like to emphasize the great practical importance of controlling diffusion rates. Periodic spatial gradients and time-periodic fields can be encountered very commonly. Their realization in practice is often quite easy so that the methods above are very applicable. Systems described by Eq. (1) are extensively investigated in science, such as in condensed matter physics for charged particles moving in thermal surfaces with periodic potentials subject to certain time-varying fields, or in biology, for finger movements [17], etc. Thus, the effects revealed in this Letter, the associated approaches for accelerating diffusion, and even the parameter regimes for effective diffusion, may be of help for designing appropriate devices or for adjusting the stochastic processes to meet particular requirements.

[1] M. O. Magnasco, Phys. Rev. Lett. **71**, 1477 (1993); **72**, 2656 (1994).
[2] R. D. Astumian and M. Bier, Phys. Rev. Lett. **72**, 1766 (1994).
[3] C. R. Doering, W. Horsthemke, and J. Riordan, Phys. Rev. Lett. **72**, 2984 (1994).
[4] R. Bartussek, P. Hänggi, and J. G. Kissner, Europhys. Lett. **28**, 459 (1994).
[5] R. Mannella, in *Noise in Nonlinear Dynamical Systems,* edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, 1989), Vol. 3.
[6] H. Risken, *The Fokker-Planck Equation,* Springer Series in Synergetics Vol. 18 (Springer-Verlag, Berlin, 1984).
[7] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, 453 (1981).
[8] C. Nicolis and G. Nicolis, Tellus **33**, 225 (1981).
[9] S. Fauve and F. Heslot, Phys. Lett. **97A**, 5 (1983).
[10] B. McNamara, K. Wiesenfeld, and R. Roy, Phys. Rev. Lett. **60**, 2626 (1988).
[11] B. McNamara and K. Wiesenfeld, Phys. Rev. A **39**, 4854 (1989).
[12] L. Gammaitoni, F. Marchesoni, E. Menichella-Saetta, and S. Santucci, Phys. Rev. Lett. **62**, 349 (1989).
[13] T. Zhou and F. Moss, Phys. Rev. A **41**, 4255 (1990).
[14] G. Hu, G. Nicolis, and C. Nicolis, Phys. Rev. A **42**, 2030 (1990).
[15] P. Jung and P. Hänggi, Phys. Rev. A **44**, 8032 (1991).
[16] M. I. Dykman, R. Mannella, P. V. E. McClintock, and N. G. Stocks, Phys. Rev. Lett. **68**, 2985 (1992).
[17] H. Haken, C. E. Peper, P. J. Beek, and A. Daffertshofer, Physica (Amsterdam) **90D**, 179 (1996).