First-Order Phase Transition in a Model for Earthquakes

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A simple mechanical model for earthquake dynamics with a generic velocity-dependent friction is investigated. It is shown that in the limit of slow driving the system undergoes a *discontinuous* (first-order) transition from stick-slip behavior to creep motion as the friction parameter is varied. This result is robust in that it does not rely on any particular choice of the friction law. The implications of these findings for the Burridge-Knopoff spring-block model for earthquakes is also discussed. In particular, it is argued that such models do not display critical behavior. [S0031-9007(96)00466-8]

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The recent suggestion [1] that earthquakes may be regarded as prototypical "self-organized critical" systems has spurred great interest in earthquake dynamics within the physics community. According to the tenets of self-organized criticality, slowly driven, spatially extended systems can display power-law behavior analogous to the static correlations occurring at critical points. In the context of earthquakes, the Gutenberg-Richter law [2], describing the distribution of energy released as a power law of the energy, has been interpreted as an evidence of self-organized criticality.

A crucial ingredient in earthquake dynamics is the friction acting at the boundary between two tectonic plates. It is indeed the velocity-weakening effect of the friction that is responsible for the sudden slip once the accumulated stresses (due to plate tectonics) overcome the static friction along the geological fault. Thus any realistic earthquake model must, of necessity, incorporate this effect. We will see in this paper, however, that a negative sensitivity of the friction with the sliding velocity, although necessary, is not sufficient for unstable sliding. More precisely, it will be shown below that the rate of decrease of the friction with the velocity must exceed a critical value in order for stick-slip motion to take place. Otherwise the motion is quasicontinuous (creep), in which case the fault would remain forever seismologically inactive. Moreover, the transition from stick-slip motion to creep is found to be of first order (i.e., discontinuous), and it will thus be argued that the dynamics of a geological fault is unlikely to display critical behavior.

Here I concentrate on the simplest model for earthquakes (see Fig. 1). In this model a block of mass m is connected by a spring of constant k to a rigid pulling rod that moves at a small constant velocity V. The block rests upon a stationary surface, which provides a velocity-dependent frictional force F that impedes the motion of the block. When the force due to the spring exceeds the threshold friction F_0, the block is set into motion; the corresponding equation of motion is

$$m\ddot{X} = k(Vt - X) - F(\dot{X}), \tag{1}$$

where $X(t)$ is the position of the block and the dots indicate time derivatives. Owing to the velocity-weakening effect of the friction, the block will undergo a rapid motion ("earthquake"), during which most of the accumulated stress is released. Then follows a quiescent period until the spring is again fully stretched in the forward direction and the cycle repeats. If the block slides by an amount Δ during one of such events, then the next earthquake will happen at a time $T = \Delta/V$. This is, of course, the recurrence time between characteristic events predicted by the classical *elastic rebound theory* [3]. It will be noted below, however, that a more "accurate" prediction for T includes a logarithmic correction in V. The model above is admittedly a crude representation of earthquakes. It has, however, the great advantage of being analytically tractable, so that one hopes that a thorough understanding of such a simple model might in turn shed further light onto the basic principles governing real earthquakes. This paper aims precisely at that.

I shall for convenience write the friction force as

$$F(\dot{X}) = F_0\Phi(\dot{X}/V_f), \tag{2}$$

where V_f is a characteristic velocity for the friction and $\Phi(x)$ is assumed to be a continuous function for $x \geq 0$ satisfying the conditions

$$\Phi(0) = 1 \quad \text{and} \quad \Phi'(0) = -1\,. \tag{3}$$

Here the prime denotes differentiation with respect to the argument. The second condition in (3) simply expresses the velocity-weakening effect of the friction, since it

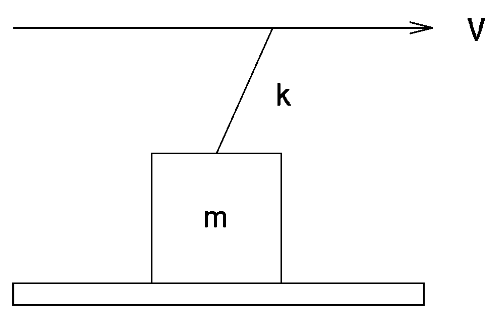

FIG. 1. Spring-block model for earthquakes.

implies that $F(\dot{X})$ will be a decreasing function of the block velocity $\dot{X}$, at least in a neighborhood of the origin. It should also be noted at this stage that the assumption implied by (2), namely, that the friction is a function of the velocity *only,* does not always hold in practice. Other possible friction models, such as slip-weakening models and rate and state dependent friction laws, have been proposed for faults [4]. I anticipate, however, that the (qualitative) results of the following analysis will not depend on the global features of the friction law (2) and should in principle be extendible to more realistic friction laws.

Before proceeding with the analysis, it is convenient to introduce dimensionless variables

$$U = \frac{k}{F_0} X, \qquad \hat{t} = \left[\frac{k}{m}\right]^{1/2} t, \tag{4}$$

so that the equation of motion (1) takes the dimensionless form

$$\ddot{U} = \nu\hat{t} - U - \Phi(\dot{U}/\nu_f), \tag{5}$$

where $\nu = V/V_0$, $\nu_f = V_f/V_0$, $V_0 = F_0/\sqrt{mk}$, and dots now represent derivatives with respect to the scaled time $\hat{t}$. (Hereupon I will drop the hat notation with the understanding that all times are measured in dimensionless units.) The velocity scale V_0 corresponds to the maximum velocity attained by a block that experiences no (kinematic) friction as it moves. The dimensionless parameters ν_f and ν are, respectively, the friction characteristic velocity and the pulling speed measured in this scale. Clearly, for this model to be relevant for real earthquakes ν must be taken very small. Indeed, during an earthquake the relative velocity between the two sides of the fault is of order m/s, while the typical relative plate velocity is of order cm/yr, so that in practice ν can be as small as 10^{-9}.

Accordingly, the main goal of this paper is to study the model above in the limit of *vanishing* pulling speed. I will show below that in the limit $\nu \to 0$ the system undergoes a *discontinuous* phase transition as ν_f crosses the critical value $\nu_f = 1/2$. Physically, these two "phases" correspond to a stick-slip motion for $\nu \le 1/2$ and a quasicontinuous motion (creep) for $\nu_f > 1/2$.

I start the analysis by considering first the linearized version of the equation of motion. In view of (2) and (3), the linearization of (5) yields

$$\ddot{U} - 2\alpha\dot{U} + U = \nu t\,, \tag{6}$$

where for convenience I have introduced the parameter $\alpha = 1/2\nu_f$. I have also redefined the origin of displacements so as to eliminate the unit constant that would otherwise appear on the right-hand side of (6). Clearly, the linear approximation above will be valid only if the block velocity is small compared to the friction characteristic velocity, i.e., $\dot{U} \ll \nu_f$. The advantage, however, is that Eq. (6) together with the initial conditions $U(0) = \dot{U}(0) = \ddot{U}(0) = 0$ can be easily solved. Here there are two cases to consider: (i) $\nu_f > 1/2$ and (ii) $\nu_f \le 1/2$.

Case 1: $\nu_f > 1/2$.—In this case $\alpha < 1$ so that the solution to (6) reads

$$U(t) = \nu\left\{e^{\alpha t}\left[\frac{2\alpha^2 - 1}{\omega}\sin\omega t - 2\alpha\cos\omega t\right] + t + 2\alpha\right\}, \tag{7}$$

where $\omega = \sqrt{1 - \alpha^2}$. The maximum velocity $\dot{U}_{\max}$ attained by the block is

$$\dot{U}_{\max} = \nu(1 + e^{\pi\alpha/\omega}), \tag{8}$$

as one can easily verify from (7). One then sees that it is always possible to choose a value of ν sufficiently small so that $\dot{U}_{\max} \ll \nu_f$. In other words, in the limit $\nu \to 0$ the linear approximation is always valid if $\nu_f > 1/2$. In this case, the block will never "feel" the nonlinear part of the friction and hence its motion is completely described by (7). From this equation one readily obtains that the block will come to a stop at the time $t = t_0$ given by the solution to the equation

$$\frac{\alpha}{w}\sin\omega t - \cos\omega t + e^{-\alpha t} = 0\,. \tag{9}$$

Thus, the duration t_0 of a "slip event" in this case is determined solely by the friction parameter ν_f and does not depend on the pulling speed ν. Using (9) into (8) one then finds that the block displacement $\Delta \equiv U(t_0)$ after such a slip event is given by

$$\Delta = \nu(\sqrt{e^{2\alpha t_0} - \omega^2} + t_0 - \alpha). \tag{10}$$

Since t_0 does not depend on ν, it then follows that as $\nu \to 0$ the displacement Δ *vanishes* whenever $\nu_f > 1/2$. Next I investigate the situation when $\nu_f \le 1/2$.

Case 2: $\nu_f \le 1/2$.—Here $\alpha \ge 1$ and the solution to (6) is given by

$$U(t) = \nu\left\{e^{\alpha t}\left[\frac{2\alpha^2 - 1}{\omega}\sinh\omega t - 2\alpha\cosh\omega t\right] + t + 2\alpha\right\}, \tag{11}$$

where $\omega = \sqrt{\alpha^2 - 1}$. Since the velocity $\dot{U}(t)$ is now a monotonously increasing function of time, the block will eventually reach velocities comparable to the friction velocity ν_f, at which point the linear approximation is no longer valid. In other words, when $\nu_f \le 1/2$ the block will always probe the nonlinear part of the friction law, no matter how small the pulling speed ν. In order to investigate the behavior of the system further it is thus necessary to consider specific models for the friction law.

Several friction models have been recently considered in the literature [5–7]. In what follows, however, rather than to be concerned with the choice of a realistic friction law, I will consider for simplicity a piecewise linear model for which analytical results can be easily obtained.

More precisely, I consider the following friction model [7]:

$$\Phi(x) = \begin{cases} 1 - x & \text{for } 0 \le x \le 1\,, \\ 0 & \text{for } x \ge 1\,. \end{cases} \tag{12}$$

As discussed above, here it is necessary to study only the case $\nu_f \le 1/2$. (Recall that if $\nu_f > 1/2$ then $\Delta \to 0$ as $\nu \to 0$, regardless of the nonlinear features of the friction law.) In this case, the solution for the block motion can be divided into three parts as follows.

Initially, when $\dot{U} < \nu_f$ the motion of the block is confined to the linear part of the friction and hence the solution is given by (11). This solution is valid until the time t_1, where $\dot{U}(t_1) = \nu_f$, after which the block enters the nonlinear regime of the friction law, here represented by a frictionless region. In the limit $\nu \to 0$ this time t_1 diverges logarithmically with ν,

$$t_1 = (\alpha - \omega)\ln\left[\frac{\omega(\alpha + \omega)}{\alpha\nu}\right], \tag{13}$$

as one can easily verify from (11). Although the block spends a "very long" period of time in this linear regime, one can easily convince oneself that during most of this time the block is essentially at rest. In other words, considerable motion will occur only for times close to the instant t_1. It is therefore convenient to introduce the renormalized time $\tau = t - t_1$. The block position in terms of τ can now be obtained by using (13) into (11) and taking the limit $\nu \to 0$. After some simplifications one finds

$$U(\tau) = \nu_f(\alpha - \omega)e^{(\alpha+\omega)\tau}, \tag{14}$$

which is valid for $\tau \le 0$. One thus sees that in terms of the renormalized time τ the solution is independent of ν; that is, the relation (14) is an *exact* result in the limit $\nu \to 0$.

For $\tau > 0$ the motion consists of two parts. First, the block will "swing" frictionlessly until the time (denoted by τ_1) at which its velocity is again equal to ν_f. Afterwards, the block will experience once more a nonzero (linear) friction until it finally stops at some later time τ_2. Setting $\nu = 0$ in (5) (since we are interested in the limit $\nu \to 0$) and using (12), one can easily solve for the motion corresponding to these two regions. Combining this with (14) one obtains the complete solution for the block position,

$$U(\tau) = \begin{cases} \nu_f(\alpha - \omega)\,e^{(\alpha+\omega)\tau} & \text{for } \tau \le 0\,, \\ \nu_f[\sin\tau - (\alpha + \omega)\cos\tau + 1] & \text{for } 0 \le \tau \le \tau_1\,, \\ \nu_f e^{\alpha(\tau-\tau_1)}[(3\alpha + \omega)\cosh\omega(\tau - \tau_1) & \\ -(\alpha + 3\omega + \frac{2}{\omega})\sinh\omega(\tau - \tau_1)] & \text{for } \tau_1 \le \tau \le \tau_2, \end{cases} \tag{15}$$

where the times τ_1 and τ_2 are given by

$$\tau_1 = 2\arctan(\alpha + \omega), \tag{16}$$

$$\tau_2 = \tau_1 + \frac{1}{2\omega}\ln\left(1 + \frac{\omega}{\alpha}\right). \tag{17}$$

The block displacement $\Delta = U(\tau_2)$ after an earthquake can now be calculated by inserting (17) into (15) and performing a straightforward if somewhat tedious algebra. Here I simply quote the final result

$$\Delta = \begin{cases} (1 + \frac{\omega}{\alpha})^{(1/2)(1+\alpha/\omega)} & \text{for } \nu_f \le \frac{1}{2}\,, \\ 0 & \text{for } \nu_f > \frac{1}{2}\,, \end{cases} \tag{18}$$

where I also collected the aforementioned result that Δ vanishes for $\nu_f > 1/2$ (as $\nu \to 0$).

One then sees that at $\nu_f = 1/2$ the system undergoes a "phase transition" in the sense that Δ vanishes for $\nu_f > 1/2$ while it takes finite values for $\nu \le 1/2$. (A plot of Δ vs ν_f is shown in Fig. 2.) Notice, however, that this transition is of a "first-order" nature, since the "order parameter" Δ changes *discontinuously* at the critical value $\nu_f = 1/2$. (I remark parenthetically that a true transition occurs only in the limit $\nu \to 0$; a finite value of ν will, of course, smooth out this transition.)

Although the model (12) is clearly too simplistic to describe actual frictional sliding, the qualitative behavior observed in this model is, notwithstanding, quite general. For instance, any friction model satisfying the conditions (2) and (3) will exhibit a *discontinuous* transition at $\nu_f = 1/2$, with stick-slip motion occurring only for $\nu_f \le 1/2$. In this case, the behavior of the system can be qualitatively divided into three regions corresponding to the three time intervals given in (15). First there is a long interval ($\tau \le 0$) of slow motion followed by a sudden slip ($0 \le \tau \le \tau_1$), where most of the actual displacement takes place, after which the block experiences again a

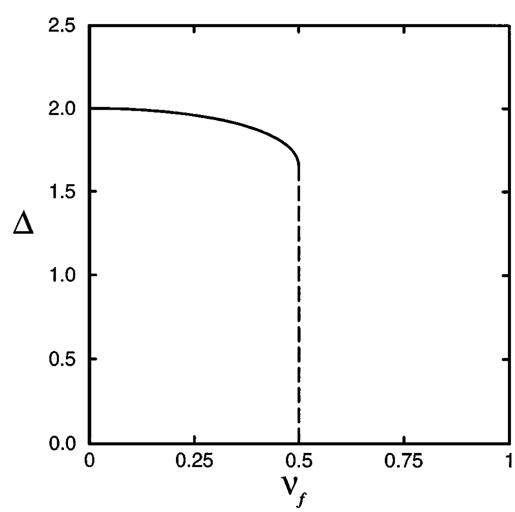

FIG. 2. The block displacement Δ vs the friction characteristic velocity ν_f for the model given in (12).

short period ($\tau_1 \leq \tau \leq \tau_2$) of linear friction until it finally comes to a stop. The time $\tau = 0$ (or $t = t_1$ in nonrenormalized units) can thus be taken as the effective "beginning" of an earthquake. In this sense, the actual time T between characteristic events will be given by $T = \Delta/\nu + t_1$. In view of (13), this implies a logarithmic correction (in ν) to the recurrence time predicted by the *elastic rebound theory,* as advertised earlier.

As already mentioned, the existence of a discontinuous transition at $\nu_f = 1/2$ does not rely on any particular choice of friction model. On the other hand, the specific details of this transition (e.g., the shape of the curve Δ vs ν_f) are obviously model dependent. For instance, in Fig. 3 I show the quantity Δ as a function of ν_f for the Carlson-Langer model [5], in which the friction force is described by the function

$$\Phi(x) = \frac{1}{1 + x}. \tag{19}$$

In this case, analytical results are not known (if possible at all), and one must resort to numerical solutions. Accordingly, for each value of ν_f in Fig. 3, the displacement Δ was computed numerically for $\nu = 10^{-9}$. Because of this finite value of ν the transition appears to be continuous in this figure. (An extrapolation for $\nu \to 0$ shows that the transition is indeed discontinuous: $\Delta = 0.17$ at $\nu_f = 1/2$, whereas $\Delta = 0$ for $\nu_f > 1/2$.)

The results above for a one-block system are also relevant for the Burridge-Knopoff spring-block model for earthquakes [8]. A homogeneous version of this model, introduced by Carlson and Langer [5], has been extensively studied by several authors [7–12]. In the context of this multiblock model, the transition from stick-slip motion to creep was first observed by de Sousa Vieira, Vasconcelos, and Nagel [10]. These authors showed that in the region corresponding to stick-slip motion, the system displays a relaxation-oscillation behavior analogous to a first-order transition. They also observed that as the parameter ν_f approached the transition value the loading-unloading hysteresis became less dramatic. They then likened this behavior to a line of first-order transitions ending at a critical point, thus suggesting that criticality (i.e., power-law behavior) in the Burridge-Knopoff model could be attained (if at all) by tuning the friction parameter to the critical value [13].

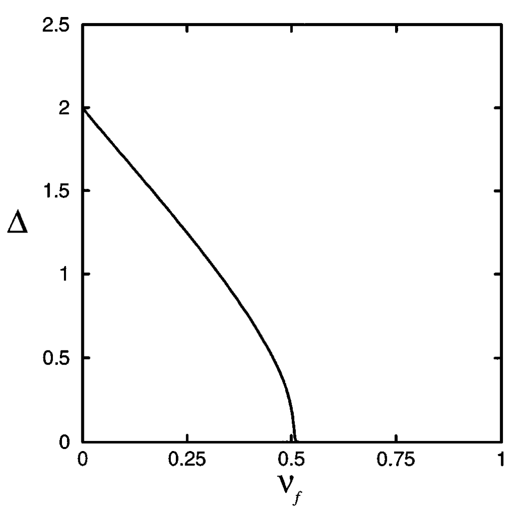

FIG. 3. Same as in Fig. 2 for the friction model (19).

The results reported in this paper demonstrate, however, that the transition from stick-slip motion to creep is *discontinuous.* In connection with the discussion in the preceding paragraph, this implies that the hysteresis mentioned above will persist (albeit less pronounced) all the way up to the transition point. In other words, (homogeneous) spring-block models cannot be brought to a critical state displaying scaling behavior [14]. I conclude thus by pointing out that if one assumes that the Burridge-Knopoff model gives a qualitatively good description of real earthquakes, it then appears that fault slipping is *not* a critical phenomenon (self-organized or otherwise).

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[14] The "scaling regime" observed in Ref. [5] was limited to small events; see, e.g., [10].