Correlation Functions in a Corner-Shaped Ising Model

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(Received 27 June 1995)

We obtain exact results at all temperatures for the two-point function of surface spins near the corner of a two-dimensional rectangular Ising ferromagnet; at the critical point, the functional form predicted by conformal theory is recovered. We are also able to calculate exactly a large class of correlation functions involving spins off the surface, and to interpret the results geometrically. [S0031-9007(96)00425-5]

PACS numbers: 75.10.Hk

Since the pioneering work of McCoy and Wu [1] on the behavior of spins on the edge of a planar Ising ferromagnet, it has been fully appreciated that surfaces have critical behavior strongly distinguished from that of the bulk [2]. One of their key contributions was to obtain the spontaneous magnetization m_e of an edge spin, and the pair correlation function at all temperatures. An essential requirement in these calculations is the translational invariance of the lattice along the edge direction: This is achieved by wrapping the lattice on a cylinder. In the field-theoretical formulation in terms of free fermions, correlation functions are evaluated by Wick's theorem [3] in terms of contractions, which the boundary conditions make translationally invariant, and which become the elements of a *Töplitz* determinant [4]. Since then, one of the unsolved problems was the correlation of spins near a corner of the lattice; not only are the determinants non-Töplitz, but one could not even calculate the contractions. In this Letter, we use a new method which allows us to treat spins on the edges at an arbitrary distance from the corner, as well as energy densities anywhere in the bulk, at all temperatures, thus extending McCoy and Wu's results.

Meanwhile, the problem acquired added piquancy from the conformal-theoretical predictions of Cardy [5] which should apply to a continuum Ising field theory at the critical point. Conformal ideas predict critical exponents and the functional form of correlation functions up to a factor, and they relate such functional forms in different lattice geometries. In this case, a conformal transformation relates the critical pair correlation in a sector with opening angle θ to the lattice-continuous limit of the exact results of McCoy and Wu for spins on the edge of an infinite cylinder, this time without an arbitrary factor. The corner magnetization exponent, now known exactly for $\theta = \pi/2$ [6], is obtained by applying scaling ideas to the pair function; only its asymptotic behavior is probed. Checking the complete predicted functional form is therefore a far more exacting test, which we amplify by giving precise formulas for the Ising lattice theory, thus allowing a check of the accuracy of the conformal theory as an approximation to the *lattice* theory. We shall show that this agreement is not exact with one spin in the corner; because, even though the power law of the decay as the other spin goes towards infinity is exact, the amplitude is not, and, moreover, it depends on the lattice anisotropy.

We consider a rectangular Ising ferromagnetic lattice with spins $\sigma_{i,j} = \pm 1$ located at sites (i,j) with $1 \le i \le N$ and $1 \le j \le M$. A configuration denoted $\{\sigma\}$ of such spins on the lattice has energy

$$E_{N,M}(\{\sigma\}) = -J_1 \sum_{i=1}^{N-1} \sum_{j=1}^{M} \sigma_{i,j}\sigma_{i+1,j} - J_2 \sum_{i=1}^{N} \sum_{j=1}^{M-1} \sigma_{i,j}\sigma_{i,j+1} \,. \tag{1}$$

For the canonical ensemble at inverse temperature β, the column-to-column transfer matrix contains two factors: T_1, accounting for Boltzmann factors between columns, and T_2, a diagonal matrix accounting for those within a column. On a Hilbert space which is the direct product of M spin-1/2 Hilbert spaces, we construct the operators V_1 and V_2 having T_1 and T_2 as their matrix representatives up to a factor

$$V_1 = \exp\left(-K_1^* \sum_{j=1}^{M} \sigma_j^z\right), \qquad V_2 = \exp\left(K_2 \sum_{j=1}^{M-1} \sigma_j^x \sigma_{j+1}^x\right), \tag{2}$$

where $K_1 = \beta J_1$, $K_2 = \beta J_2$, the dual coupling constant K_1^* is defined by $\exp(-2K_1^*) = \tanh K_1$, and the representation used has σ_j^x diagonal for $1 \le j \le M$.

For the present problem the symmetrization $V' = V_1^{1/2} V_2 V_1^{1/2}$ is convenient. It can be diagonalized using the Jordan-Wigner transformation to fermions $f_j = \frac{1}{2} P_{j-1}(\sigma_j^x - i\sigma_j^y)$ with $P_0 = 1$, and $P_j = \prod_{k=1}^{j}(-\sigma_k^z)$ for $1 \le j \le M$, and the spinors defined by $\Gamma_{2j-1} = f_j^\dagger + f_j$ and $\Gamma_{2j} = -i(f_j^\dagger - f_j)$. The Euclidean equation of motion is linear, and we define a matrix R such that $V'\Gamma^T V'^{-1} = \Gamma^T R$. The eigenvectors of R have a special significance: let $Ry_k = e^{\gamma(k)} y_k$ and $Ry_k^* = e^{-\gamma(k)} y_k^*$ with

$\gamma(k) \geq 0$ (this structure is mandatory); then if we define

$$X_k = \sum_{m=1}^{2M} y_{m,k} \Gamma_m \tag{3}$$

with $\| y_k \|^2 = 1/2$, the X_k are fermions and

$$V' = \exp\left(- \sum_k \gamma(k) \left(X_k^\dagger X_k - \frac{1}{2} \right) \right). \tag{4}$$

It turns out that

$$\cosh\gamma(k) = \cosh 2K_1^* \cosh 2K_2 - \sinh 2K_1^* \sinh 2K_2 \cos k \tag{5}$$

and that there is a quantization condition $e^{iMk} = -i\alpha(k) e^{i\delta^*(k)}$ where $\alpha(k) = \pm i$, and the function δ^* is defined in [7] along with δ', φ_0, φ_1 and the expression of $y_{m,k}$. Note that all these functions have a similar analytic structure, with branch points at $A^{\pm 1}$, $B^{\pm 1}$ with $A = \coth K_1^* \coth K_2$ and $B = \coth K_1^* \tanh K_2$. The appearance of α can be understood by appealing to the reflection invariance of V' [7].

The vacuum for the X_k is denoted $|\Phi\rangle$. This state is the maximum eigenvector of V', is nondegenerate for any temperature (provided M is finite), and, since V' is invariant under parity and reflection, $|\Phi\rangle$ is an eigenstate of P and Σ; the corresponding eigenvalues are conserved by continuity so we can determine them by taking $T \to \infty$, in which case V' goes to V_1, thus $|\Phi\rangle$ goes to the state $|0\rangle$ representing a column of free spins which is even and satisfies $\Sigma|0\rangle = |0\rangle$ therefore at any temperature $P|\Phi\rangle = |\Phi\rangle$ and $\Sigma|\Phi\rangle = |\Phi\rangle$.

There are M values of k giving nontrivially different y_k. If M is big enough, above the critical temperature they are all real, but below the critical temperature there is one mode with a pure imaginary wave vector $k = i\ln B + O(B^{-2M})$ giving asymptotic degeneracy in the spectrum, $\gamma(k) = O(B^{-M})$. We denote X_c the corresponding fermion operator; its reflection behavior is $\alpha_c = i$.

We consider the correlation function between two points on the same edge (perpendicular to transfer direction) $g(j_1, j_2) = \langle \sigma_{1,j_1} \sigma_{1,j_2} \rangle$. Taking $N \to \infty$, we obtain for $j_1 < j_2$

$$g(j_1, j_2) = e^{2K_1^*} \frac{\langle 0 | f_{j_2} f_{j_1} | \Phi \rangle}{\langle 0 | \Phi \rangle}. \tag{6}$$

The expansion of f_j in terms of the creation and annihilation operators $X_k^\dagger$ and X_k, using the inversion of (3) and its adjoint, reduces the problem to the determination of the matrix elements $\langle 0 | X_{k_1}^\dagger X_{k_2}^\dagger | \Phi \rangle$. A selection rule is associated with the reflection symmetry: These matrix elements are nonzero only if $X_{k_1}^\dagger$ and $X_{k_2}^\dagger$ have different behaviors. For k_1 and k_2 real we define

$$K_M(k_1, k_2) = \frac{e^{-i\varphi_0(k_1)} + e^{-i\varphi_1(k_1)}}{N_{k_1}} \frac{e^{-i\varphi_0(k_2)} + e^{-i\varphi_1(k_2)}}{N_{k_2}} \times \frac{\langle 0 | X_{k_1}^\dagger X_{k_2}^\dagger | \Phi \rangle}{\langle 0 | \Phi \rangle}. \tag{7}$$

We denote $z_1 = e^{ik_1}$, $z_2 = e^{ik_2}$, etc., and use k or the corresponding z with indifference as argument in any function of the wave vector. Let $K^{(-+)}(z_1, z_2)$ be the thermodynamic limit of (7) in the case $\alpha(k_1) = -i$, $\alpha(k_2) = i$.

We now obtain an integral equation for the matrix elements by using the vacuum conditions $\langle 0 | f_j^\dagger = 0$ which characterize a column of free spins. If we write $0 = \langle 0 | f_j^\dagger X_{k_2}^\dagger | \Phi \rangle$ and expand the $f_j^\dagger$ linearly in terms of X_k and its adjoint using the inversion of (3) again, we get a linear equation relating the desired $\langle 0 | X_{k_1}^\dagger X_{k_2}^\dagger | \Phi \rangle$ to $\langle 0 | \Phi \rangle$, and thus an equation for the K of (7). For any $z_2 = e^{ik_2}$ on the unit circle satisfying $e^{iMk_2} = e^{i\delta^*(k_2)}$:

$$\frac{1}{4M} \sum_{k=-\pi}^{k=\pi} z^j K(z, z_2) = -i \sin\delta'(z_2) \left(z_2^{j-1} e^{-i\delta^*(z_2)} - z_2^{-j} \right). \tag{8}$$

We caution the reader that it may not be possible to get any solution, let alone a unique one in the thermodynamic limit. For the similar problem where k_2 is replaced by the imaginary wave vector our first solution [6,7] involved the inverse of the singular integral operator Y of Yang [8] which is known not to exist for $T < T_c$ *because there is a null spectrum.* But it turned out that the *statistical-mechanical* dual of Y was needed, for which the inverse does exist [9]. Such a method would probably work for this case, but we will use the much more amenable Wiener-Hopf method which follows.

Taking any z_1 such that $|z_1| > 1$, multiplying (8) by z_1^{-j}, and summing on j gives, in the thermodynamic limit,

$$\frac{1}{2i\pi} \int_{|t|=1} \frac{dt}{t - z_1} K^{(-+)}(t, z_2) = -4i \sin\delta'(k_2) \left(\frac{1}{z_1 z_2 - 1} - \frac{e^{-i\delta^*(k_2)}}{z_1 - z_2} \right). \tag{9}$$

The second ingredient in the Wiener-Hopf method of [4,10] as used in [7] is the relationship between opposite values of k which we read from the expression of $X_k^\dagger$:

$$K(k_1, k_2) = e^{-ik_1 - i\delta^*(k_1)} K(-k_1, k_2) = e^{-ik_2 - i\delta^*(k_2)} K(k_1, -k_2). \tag{10}$$

The difference with the method used in [7] is that we now have poles on the unit circle on the right hand side of (9). However, (9) is equivalent to saying that

$$K_+^{(-+)}(z_1, z_2) = K^{(-+)}(z_1, z_2) - 8i \sin\delta'(k_2) \left(\frac{1}{z_1 z_2 - 1} - \frac{e^{-i\delta^*(k_2)}}{z_1 - z_2} \right) \tag{11}$$

is analytic for z_1 inside or on the unit circle, and we can now use (10) and the Wiener-Hopf method, giving below the critical temperature

$$K_+^{(-+)}(z_1, z_2) = \frac{4(AB - 1)(1 + z_1)(1 + z_2)}{(z_1 - z_2)(z_1 z_2 - 1)} \frac{(1 - z_2)^2}{(Bz_2 - 1)\sqrt{(z_2 - B)(z_2 - A)}} \sqrt{\frac{z_1 - B}{z_1 - A}} . \tag{12}$$

If we now consider the supercritical case, the integral equation (9) and the symmetry condition (10) do not permit us to obtain a *unique* solution, and, even if we take into account the antisymmetry of the matrix element in the exchange of k_1 and k_2 (as in [11]), we are still left with an arbitrary constant. To obtain unicity, we require the correlation function to vanish for infinite separations (i.e., disallow any spontaneous magnetization for $T > T_c$), giving finally

$$K_+^{(-+)}(z_1, z_2) = \frac{4(1 - AB)(1 - z_2^2)(1 - z_1^2)}{(z_1 - z_2)(z_1 z_2 - 1)\sqrt{(z_1 - A)(Bz_1 - 1)(z_2 - A)(Bz_2 - 1)}} . \tag{13}$$

We now use spectral decomposition of both fermion operators appearing in (6) to get the correlation function $g(j_1, j_2)$. For subcritical temperatures, it contains three terms: the first two involve $X_c^\dagger$ and have a structure similar to the spontaneous edge magnetization near the corner [6,7], and the third, which is also the only one for supercritical temperatures, is

$$-\frac{P}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{ij_1k_1} e^{ij_2k_2} K^{(-+)}(k_1, k_2)}{(1 + e^{-i\delta'(k_1)})(1 + e^{-i\delta'(k_2)})} \, dk_1 \, dk_2 . \tag{14}$$

In this two-dimensional Fourier transform the only factors coupling the two integration variables are $(z_1 - z_2)^{-1}$ and $(z_1 z_2 - 1)^{-1}$ coming from (12) or (13). This suggests the introduction of a "lattice second derivative"

$$\begin{aligned} g''(j_1, j_2) = {} & g(j_1 + 2, j_2 + 1) - g(j_1 + 1, j_2 + 2) \\ & - g(j_1 + 1, j_2) + g(j_1, j_2 + 1) , \end{aligned} \tag{15}$$

which indeed turns out to be the product of a function of j_1 and a function of j_2. We now consider the critical point, i.e., $B \to 1$. The first two terms in $g(j_1, j_2)$ vanish if we approach the critical temperature from below, and in any case $g''(j_1, j_2)$ is given by

$$\begin{aligned} g''(j_1, j_2) = {} & \frac{2e^{2K_1^*}}{\pi^2(1 - A)} \left[\int_{A^{-1}}^1 z^{j_1 - 1} \, dz \sqrt{(Az - 1)(1 - z)} \right] \\ & \times \left[\int_{A^{-1}}^1 z^{j_2 - 1} \, dz \sqrt{(Az - 1)(1 - z)} \right] . \end{aligned} \tag{16}$$

This is not yet identical to the conformal prediction: Indeed, to compare our results to those of conformal field theory, we need to take the lattice-continuous limit of $g(j_1, j_2)$. Let $y_1 = aj_1$ and $y_2 = aj_2$, where a is the lattice spacing. Then, taking $a \to 0$ and $j_1, j_2 \to \infty$ with fixed y_1 and y_2 in (15) and (16) gives

$$\frac{\partial^2 G}{\partial y_1^2} - \frac{\partial^2 G}{\partial y_2^2} = -\frac{e^{2K_1^*}}{2\pi} y_1^{-3/2} y_2^{-3/2} , \tag{17}$$

where the lattice-continuous limit is defined by

$$G(y_1, y_2) = \lim_{a \to 0} a^{-1} g(a^{-1} y_1, a^{-1} y_2) . \tag{18}$$

We define the auxiliary variables $u = y_2 + y_1$ and $v = y_2 - y_1$ in order to solve (17); rather than using boundary conditions, we extend the domain of the definition of G to benefit from its oddness in v [obvious from the definition (6)] and u [which we see by using (10) in (14) and $\delta' = \delta^*$ which is specific to the critical point, giving $g(1 - j_1, 1 - j_2) = g(j_1, j_2)$ and therefore $G(-y_1, -y_2) = G(y_1, y_2)$]. This gives

$$G(y_1, y_2) = \frac{2e^{2K_1^*}}{\pi} \frac{\sqrt{y_1 y_2}}{y_2^2 - y_1^2} . \tag{19}$$

The conformal result for the corner pair correlation is obtained by transformation of the lattice-continuous limit of the McCoy and Wu result [1] for spins on the edge of a cylinder,

$$\lim_{a \to 0} a^{-1} \langle \sigma(1, a^{-1} z_1) \sigma(1, a^{-1} z_2) \rangle = \frac{e^{2K_1^*}}{\pi} \frac{1}{|z_1 - z_2|} . \tag{20}$$

One aspect of conformal theory states that [5,12]

$$\langle \sigma(w_1) \sigma(w_2) \rangle_{\text{corner}} = \left| \frac{dz_1}{dw_1} \right|^x \left| \frac{dz_2}{dw_2} \right|^x \langle \sigma(z_1) \sigma(z_2) \rangle_{\text{edge}} \tag{21}$$

with $z = w^2$ and $x = 1/2$, which results in complete agreement (including the prefactors) with the exact lattice-continuous theory (19).

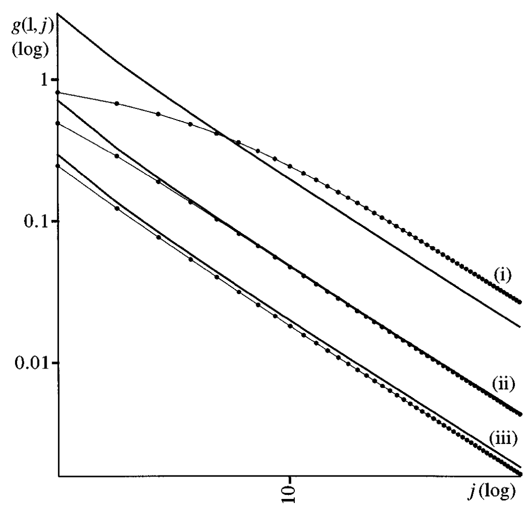

FIG. 1. Comparison of $g_{1c}(1, j)$ constructed from the conformal result (thick lines) with the exact critical correlation function $g(1, j)$ (points) obtained by numerical integration of (14) with (12) or (13) for $B = 1$ and several values of A: $A_{(i)} = 1.5$, $A_{(ii)} = 3 + 2\sqrt{2}$ [the isotropic case $K_1 = K_2 = \frac{1}{2}\ln(1 + \sqrt{2})$], and $A_{(iii)} = \infty$ (the extreme anisotropic or Hamiltonian limit $K_2 \ll K_1$).

We now consider some numerical results obtained by integration of the exact expression (14) for $g(j_1, j_2)$ at the critical point, and compare them with the results extracted in the normal way from the lattice-continuous theory (19), denoted by $g_{1c}(j_1, j_2) = aG(aj_1, aj_2)$. The conformal prediction (19) becomes exact if j_1 and j_2 are both far away from the corner and from each other, but even in the isotropic case the asymptotic behavior of the correlation function between the corner spin and a surface spin going to infinity (see Fig. 1) is $g(1, j) \propto 1.50 j^{-3/2}$, whereas (19) predicts $\lim_{j\to\infty} j^{3/2} g_{1c}(1, j) = 2(1 + \sqrt{2})/\pi \approx 1.54$. Most optimistically, we might have hoped that corrections to the conformal theory here would not be found in the leading term; the power law is certainly correct, but the amplitude is in error by roughly 2.7%. This deviation probably occurs because the corner is a singular point of the conformal transformation $z = w^2$ used in (21).

Higher-order edge-spin correlation functions or correlations containing energy densities anywhere in bulk can be studied at all temperatures by using an extension of Wick's theorem [11]: To determine a $2n$-particle matrix element, we use the vacuum property to get an equation similar to (8), except that the right-hand side now contains $2n - 1$ terms involving $z_2, z_3, \ldots, z_{2n}$ in place of z_2, each term being multiplied by a $(2n - 2)$-particle matrix element. Solving this recurrence gives the $2n$-particle matrix elements as Pfaffians in which the contraction function is the two-particle matrix element which we calculated above.

An energy density operator is turned into a pair of local fermion operators, and, for instance, a correlation function between two energy densities would involve four fermionic operators. It is interesting to look at the structure of a contraction between two operators located at points (x_1, y_1) and (x_2, y_2):

$$\int_0^{2\pi} e^{-|x_2-x_1|\gamma(k)}\left(a(k)e^{i(y_2-y_1)k} - b(k)e^{i(y_2+y_1)k}\right)dk + \int_0^{2\pi}\int_0^{2\pi} \frac{c(k_1)d(k_2)e^{-x_1\gamma(k_1)}e^{-x_2\gamma(k_2)}e^{iy_1k_1}e^{iy_2k_2}}{(e^{ik_1} - e^{ik_2})(e^{i(k_1+k_2)} - 1)}\, dk_1\, dk_2\,, \quad (22)$$

where the functions *a*, *b*, and *d* depend on the details of the contraction function being considered but are always analytic in the annulus $B^{-1} < |e^{ik}| < B$, and do not involve either (x_1, y_1) or (x_2, y_2). If we take distances large compared to the correlation length so that we are in the "quasiclassical" regime, (22) can be evaluated by a saddle point method. The first line is related only to the horizontal boundary (and not to the vertical one or to the corner) and separates in two terms: In the first one the saddle point comes from $e^{-|x_2-x_1|\gamma(k)+i(y_2-y_1)k}$, and we represent it (see Fig. 2) by a straight line (a) joining the two points, by analogy with a geometric light ray or the trajectory of a classical particle; the other one is controlled by $e^{-|x_2-x_1|\gamma(k)+i(y_2+y_1)k}$ and corresponds to the line (b) bouncing (reflecting) on the horizontal boundary. The residues (principal parts) at $k_1 = \pm k_2$ in the second line of (22) give two single integrals which correspond to the paths (c) bouncing on the vertical boundary and (d) bouncing on both boundaries. The rest of the second line remains a double integral which we evaluate as a product of contributions from the saddle points of $e^{-x_1\gamma(k_1)+iy_1k_1}$ and $e^{-x_2\gamma(k_2)+iy_2k_2}$, and therefore represent by (e) joining both points to the corner.

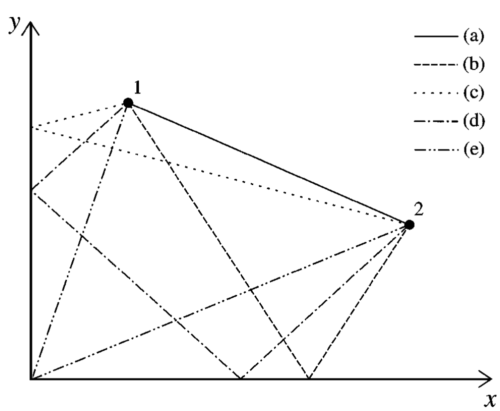

FIG. 2. Quasiclassical interpretation of the contraction (21). The meaning of lines (a)–(e) is explained in the text.

The authors thank T. Spencer and the Institute for Advanced Study very much for hospitality and excellent facilities. D. B. A. acknowledges financial support from the EPSRC and the EU under grants GR/J78044 and ERBHBICT 941666. He also thanks É. Brézin for interesting discussions and for the hospitality of the École Normale Supérieure. F. T. L. thanks J. L. Cardy, M. Howard, and M. J. E. Richardson for helpful discussions. He is very grateful to Merton College, Oxford, for the award of a Senior Scholarship, and also thanks the EPSRC for the award of a Research Studentship.

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