Glass Formation in a Periodic Long-Range Josephson Array

P. Chandra,[1] M. V. Feigelman,[2] and L. B. Ioffe[2,3]
[1]*NEC Research Institute, 4 Independence Way, Princeton, New Jersey 08540*
[2]*Landau Institute for Theoretical Physics, Moscow, Russia*
[3]*Department of Physics, Rutgers University, Piscataway, New Jersey 08855*
(Received 1 September 1995)

We present an analytic study of a dynamical instability in a periodic long-range Josephson array frustrated by a weak transverse field. This glass transition is characterized by a diverging relaxation time and a jump in the Edwards-Anderson order parameter; it is *not* accompanied by a coinciding static transition. [S0031-9007(96)00454-1]

PACS numbers: 74.50.+r

Glass formation in the *absence* of intrinsic disorder is a long-standing problem. Although vitrification is ubiquitous, a minimalist microscopic model of this phenomenon remains the subject of active discussion [1,2]. Because glass formation is a dynamical transition that is not necessarily accompanied by a static one, it lies outside the framework of the Landau theory. Furthermore, this glass transition leads to a low-temperature phase with broken ergodicity *without* the selection of a *unique* state. A successful phenomenological theory should describe the "partitioning" of phase space below the transition temperature (T_G) into an exponential number of metastable states; specifically, it should explain how the system becomes "stuck" at T_G in one of these states that is separated from the others by barriers which scale with the system size.

Unfortunately, a basic theory of glass formation has not yet been found. Several microscopic nonrandom models have been proposed; most were studied via a mapping to disordered systems [2]. Recently, possible glassiness in the absence of disorder has been studied in a periodic long-range Josephson array using a direct, analytic approach [3]; furthermore, this system may be realized experimentally. An analysis of its static behavior indicates a first-order transition into a low-temperature phase characterized by an extensive number of states separated by infinite barriers. In this Letter we continue the study of this system and show that it displays a true dynamical instability that *precedes* the static transition, as expected in a glass [1].

The proposed array is a stack of two mutually perpendicular sets of N parallel wires with Josephson junctions at each node (Fig. 1) that is placed in an external transverse field. The classical thermodynamic variables of this system are the superconducting phases associated with each wire. Here we shall assume that the Josephson couplings are sufficiently small so that the induced fields are negligible in comparison with H. We can therefore describe the array by the Hamiltonian

$$\mathcal{H} = -\sum_{m,n}^{2N} s_m^* J_{mn} s_n \,, \tag{1}$$

where J_{mn} is the coupling matrix

$$\hat{J} = \begin{pmatrix} 0 & \hat{J} \\ \hat{J}^\dagger & 0 \end{pmatrix}, \tag{2}$$

with $J_{jk} = (J_0/\sqrt{N}) \exp(2\pi i \alpha jk/N)$ and $1 \le (j,k) \le N$, where $j(k)$ is the index of the horizontal (vertical) wires; $s_m = e^{i\phi_m}$, where the ϕ_m are the superconducting phases of the $2N$ wires. Here we have introduced the flux per unit strip, $\alpha = NHl^2/\phi_0$, where l is the internode spacing and ϕ_0 is the flux quantum; the normalization has been chosen so that T_G does not scale with N.

Because every horizontal (vertical) wire is linked to every vertical (horizontal) wire, the number of nearest neighbors (z) in this model is N; we can therefore study it with a mean-field approach. For $1/N \ll \alpha < 1$ the number of low-temperature metastable solutions is extensive [3]. Furthermore, this degeneracy develops simultaneously with the instability of the paramagnetic phase; at this temperature, interactions do not favor a particular state, and, because $z \sim N$, the barriers separating these

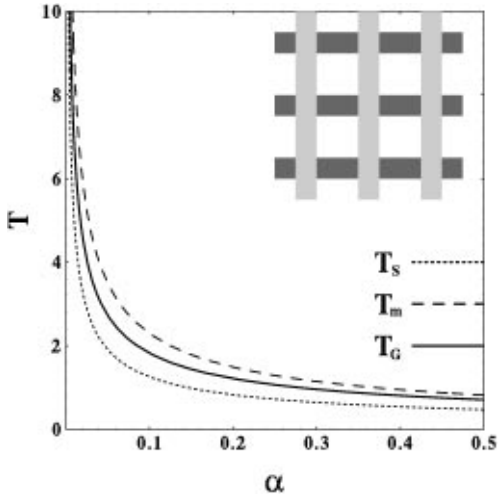

FIG. 1. The phase diagram of the array (inset), where T_G indicates the temperature associated with the dynamical instability discussed in this Letter, T_S is the speculated equilibrium transition temperature, and T_m is the "superheating" temperature where the low-temperature metastable states cease to exist.

low-temperature metastable solutions are effectively infinite. The static response displays *no* soft mode instability but indicates a jump in the Edwards-Anderson order parameter in the vicinity of $T_0 = J_0/\sqrt{\alpha}$.

Before presenting our quantitative treatment of the dynamical behavior of this array, we discuss the qualitative picture of the glass transition that emerges from our results (cf. Fig. 1). As T approaches T_m^+, where $T_m^+ \sim T_0$, there appear a number of metastable states in addition to the paramagnetic free-energy minimum; most likely they are energetically unfavorable and thus do not "trap" the system upon cooling from high temperatures. As $T \to T_G^+$, the paramagnetic minimum is "subdivided" into an extensive number of degenerate metastable states separated by effectively infinite barriers, and the system is dynamically localized into one of them. Qualitatively, in the interval $T_m > T > T_G$ there appear many local minima in the vicinity of the paramagnetic state separated by *finite* barriers; these barriers increase continuously and become infinite at $T = T_G$. Each of these minima is characterized by a finite "site magnetization" $m_i = \langle s_i \rangle_T$, where "site" refers to a wire. When $T > T_G$ thermal fluctuations average over many states so that $\langle m_i \rangle \equiv 0$. At $T = T_G$ the system is localized in one metastable state, and there is an associated jump in the Edwards-Anderson order parameter ($q = \frac{1}{N}\sum_i \langle m_i \rangle^2$). The low-temperature phase is characterized by a finite q and by the presence of a memory, $\lim_{t' \to \infty} \Delta(t, t') \neq 0$, where $\Delta(t, t')$ is the anomalous response. We expect that at $T = T_G$ the metastable states are degenerate, and thus there can be no thermodynamic selection. However, at lower temperatures, interactions will probably break this degeneracy and select a subset of this manifold; we then expect an ($t \to \infty$) equilibrium first-order transition (T_S) which should be accompanied by a jump in the local magnetization. In order to observe this transition at T_S the array must be equilibrated on a time scale (t_W) longer than that (t_A) necessary to overcome the barriers separating its metastable states; t_A scales exponentially with the number of wires in the array. Thus the equilibrium transition at T_S is observable *only* if $t_W \to \infty$ *before* the thermodynamic limit ($N \to \infty$) is taken; in the opposite order of limits, only the dynamical transition occurs.

We now begin a more quantitative analysis of the dynamic instability in this periodic array. Because our focus is on the long-time behavior of this system, we expect the details of the single-spin dynamics to be irrelevant; we therefore choose to study the simplest form, namely, that of soft spins with Langevin relaxational dynamics. More specifically, we introduce a "potential," $V(S_i) = V_0(|S|^2 - 1)^2$, at each wire which constrains the magnitude of each spin, $|S_i| \approx 1$, and assume the equations of motion

$$\tau_b \dot{S}_i = -\frac{1}{T}\frac{\partial(\mathcal{H} + V)}{\partial S_i^*} + \zeta_i\,, \tag{3}$$

$$\langle \zeta_i(t)\zeta_j(t') \rangle = 2\tau_b \delta(t - t')\delta_{ij}\,, \tag{4}$$

where τ_b is a microscopic time scale. The dynamics (3) reproduces the dynamics of the overdamped Josephson junctions with individual resistance R if $\tau_b = \hbar^2/(2e)^2 RT$ and $V_0 \to \infty$. In order to average the solution of (3) over the thermal noise, ζ, we use a generating functional. For example, the average supercurrent in the array is given by

$$\langle I_{ij} \rangle = \int I_{ij} \exp(\mathcal{A}[S, \hat{S}]\mathcal{D}S\mathcal{D}\hat{S})\,. \tag{5}$$

Here the current $I_{ij} = (2e/\hbar)\,\mathrm{Im} S_i^* J_{ij} S_j$, and the action is

$$\mathcal{A} = \int dt \Bigg[\hat{S}\left(\tau_b \dot{S} + \frac{1}{T}\frac{\partial(\mathcal{H} + V)}{\partial S^*}\right) + \tau_b \hat{S}^2 + \text{H.c.} \Bigg], \tag{6}$$

where we have not included the terms that arise from the Jacobian since they do not affect the long-time response [4–6].

We perform our calculations by resumming the terms in the $\mathcal{H}/T$ expansion of (5) which are leading order in $1/N$, this is a dynamical analog of the high-temperature series expansion previously used to study the static behavior of this array [3]. The crucial ingredients of this technique are the response $[G_{mn}(t, t') = \langle s_m(t)\hat{s}_n(t') \rangle]$ and the correlation $[D_{mn}(t, t') = \langle s_m(t) s_n^*(t') \rangle]$ functions. For $T > T_G$, these depend *only* on the time differences and thus can be related by the fluctuation-dissipation theorem

$$G_{ij}(t - t') = -\frac{\partial D_{ij}(t - t')}{\partial t}\,\theta(t - t')\,. \tag{7}$$

The leading diagrams (in $1/N$) for $G_{ij}(t - t')$ are displayed in Fig. 2(a). The presence of the "constraining potential" V in the action (6) results in finite higher-order irreducible single-site spin correlations, which play the role of interaction vertices in this diagrammatic technique. However, the corrections to the response function shown in Fig. 2(b) are small in $1/N$ in comparison with those in Fig. 2(a).

We emphasize that, as in the static case, the single-site response is renormalized; here we consider the local Green's function $[\tilde{G}(t - t')]$ that is irreducible with respect to the J_{ij} lines. Possible self-energy corrections to $\tilde{G}(t - t')$ are shown in Fig. 2(c) and will be discussed below. Summing the geometric series shown in Fig. 2(a), we obtain

$$\hat{G}_\omega = \frac{1}{\tilde{G}_\omega^{-1} - \beta^2 (J^\dagger J)\tilde{G}_\omega} \tag{8}$$

for the response function connecting wires of the same type (horizontal or vertical). The matrix $(J^\dagger J)_{ij}$ depends only on the "distance" $i - j$ and acquires a simple form

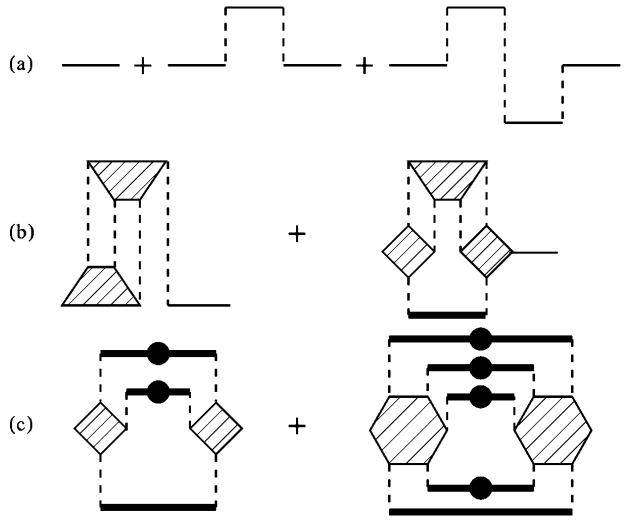

FIG. 2. Diagrammatic expansion for the response function $\hat{G}(t)$; the dashed, solid, and thick (with dot) lines are the coupling matrix J, the single-site irreducible $[G(t)]$, and the full response (correlation) functions, respectively. (a) Leading order in $1/N$. (b) Subleading order in $1/N$. (c) Leading terms in the expansion for $\tilde{G}(t)$; the first diagram dominates at small α.

in Fourier space $(J^\dagger J)_p = (J_0^2/\alpha)\theta(\alpha\pi - |\rho|)$; in this representation the Green function becomes

$$G_\omega(p) = \frac{\theta(\alpha\pi - |p|)}{\tilde{G}_\omega^{-1} - (\beta J_0)^2\tilde{G}_\omega/\alpha} + \frac{\theta(|p| - \alpha\pi)}{\tilde{G}_\omega^{-1}}. \quad (9)$$

The static limit ($\omega = 0$) of $T\tilde{G}_\omega^{-1}$ coincides with the locator, $A(T)$, discussed previously [3]; in the absence of Onsager feedback terms, $\tilde{G}_0^{-1} = 1$. Therefore we see in (9) that there would be a static instability at $\tilde{G}_0^{-1} = G_c = \beta J_0/\sqrt{\alpha}$. For $\Theta = (T - T_0)/T_0 \gg \sqrt{\alpha}$, all feedback effects are negligible. Here the time dependence of $\tilde{G}_\omega^{-1}$ is set by a microscopic time scale τ_b and $\tilde{G}_b^{-1}(\omega) = \beta A(T) - i\omega\tau_b$; inserting this $\tilde{G}_b^{-1}(\omega)$ into (9) we see that the long-time behavior of $G_\omega(p)$ is dominated by the first term which results in a long relaxation time $\tau = (2/a)\tau_b$, where $a = \beta(A - J_0^2/\alpha A) \approx 2\Theta$. In this regime the single-site response function is

$$G(t) = \frac{\alpha}{2\tau_b}\, e^{-t/\tau}, \qquad t \gg \tau_b. \quad (10)$$

At lower temperatures, $\Theta \lesssim \sqrt{\alpha}$, the feedback effects become important; they modify $\tilde{G}_0^{-1}$ so that it approaches G_c only asymptotically at $T \to 0$, and instead a first-order transition occurs [3]. The retardation of the Onsager terms is also crucial and significantly affects the long-time behavior. Qualitatively, the resulting dynamical instability, described below, is due to the time dependence of the cavity field which itself is determined by single-site susceptibilities; the time scale associated with the relaxation of $G(t)$ increases continuously due to feedback through the Onsager terms. Formally, the latter introduce an additional frequency-dependent part of the local response

$$\tilde{G}_\omega = \tilde{G}_b(\omega) + \Sigma_\omega \quad (11)$$

as a self-energy Σ_ω such that $\Sigma_0 = 0$ since we have chosen our normalization so that $\tilde{G}_0^{-1} = \beta A(T)$.

The on-site self-energy terms [see Fig. 2(c)] are the simplest for $\alpha \ll 1$, and thus we will consider this regime. We focus on the long-time response of this system which is dominated by the first term in (9) when $\tilde{G}_0^{-1} \approx G_c$; its weight is proportional to α [cf. (10)]. Thus, in the limit of $\alpha \ll 1$, the slowly decaying parts of $D(t)$ and $G(t)$ scale with α, and the dominant self-energy contribution contains the minimal number of these functions [$\Sigma^{(3)}$ in Fig. 2(c)] and is given by

$$\Sigma_\omega = \frac{3}{2}\,\Gamma^2 \int \hat{D}^2(t)\hat{G}(t)\,(e^{i\omega t} - 1)dt, \quad (12)$$

where $\hat{D}$ and $\hat{G}$ are long-time single-site correlation and response functions, respectively [7]. Here Γ is the four-spin vertex; we neglect its transient time dependence and approximate it by its static value $\Gamma = -1$ which is determined by the high-temperature single-site nonlinear susceptibility $\chi_3 = -1/T^3$. The set of Eqs. (7), (9), (11), and (12) are sufficient to determine the response and the correlation functions of the array.

Since we would like to detect a dynamical instability, we only consider the long-time behavior of the response function, i.e.,

$$\hat{G}(t) = \alpha \int \frac{e^{-i\omega t}}{a - 2(\Sigma_\omega + i\omega\tau_b)}\left(\frac{d\omega}{2\pi}\right). \quad (13)$$

Using (12), (7), and (13) we obtain a closed form equation

$$\int_0^\infty \hat{D}(t)e^{i\omega t}dt = \left(\frac{\alpha}{a}\right)\frac{2\tau_b + \int_0^\infty \hat{D}^3(t)e^{i\omega t}dt}{a - i\omega[2\tau_b + \int_0^\infty \hat{D}^3(t)e^{i\omega t}dt]} \quad (14)$$

which results in the asymptotic behavior

$$\hat{G}(t) = \frac{2\alpha}{3a\tau_R}\, e^{-t/\tau_R}, \qquad \hat{D}(t) = \frac{2\alpha}{3a}\, e^{-t/\tau_R} \quad (15)$$

as can be verified by direct substitution; here τ_R is the physical relaxation time which diverges as $a \to a_c$, where $a_c = \sqrt{2}(\alpha/3)^{3/4}$. Equation (14) is very similar to that derived for density correlations in a mode-coupling approach to the liquid-glass transition [8]; furthermore, (14) is identical to the dynamical equation obtained for the $p = 4$ (disordered) spherical model [9]. Following these previously discussions, we find the scaling form of the relaxation time

$$\tau_R = \frac{2\mu\tau_b}{a_c}\left(\frac{a_c}{a - a_c}\right)^\nu, \quad (16)$$

where $\nu = 1.765$; here the value $\mu = 4.5$ was determined numerically. We see from (16) that τ_R diverges continuously at $\Theta_G = -(3^{3/4}/2^{1/2})\alpha^{1/4}$ (cf. Fig. 1). At

$\Theta = \Theta_G$ the long-time part of $D(t)$ shown in (15) becomes constant, $q = \sqrt{2}(\alpha/3)^{1/4}$, indicating a jump in the Edwards-Anderson order. The resulting phase diagram is displayed in Fig. 1; we note that $T_G = (1 + \Theta_G)T_0$ occurs at a lower temperature than T_m, where the last low-temperature metastable states disappear [3], as discussed above.

The response function $G(t)$ is a susceptibility with respect to the field conjugate to $S = \exp(i\phi)$, and thus cannot be measured directly. However, using $G(t)$ found above, we can determine the ac response to a time-varying physical magnetic field $H(t)$ which is experimentally accessible. We focus on the total magnetic moment of the array generated by the Josephson currents:

$$\mathcal{M} = \frac{1}{2}\left(\frac{2e}{\hbar c}\right) l^2 \sum_{mn} \langle S_m \tilde{J}_{mn} S_n \rangle , \tag{17}$$

where $\tilde{J}_{mn} = imn J_{mn}$ if m and n are indices referring to horizontal and vertical wires, respectively. We would like to determine the response in this magnetization to a time-varying field; we use the fact that $\mathcal{M} = 0$ for static H to write

$$\frac{\partial \mathcal{M}(t)}{\partial H(t')} = \left(\frac{2e}{\hbar c}\right)^2 l^4 \,\mathrm{Re}\,\mathrm{Tr}\,\tilde{J}\hat{G}(t,t')\tilde{J}\hat{D}(t,t') . \tag{18}$$

In order to evaluate (18) we will need the response function connecting wires of different type (horizontal or vertical),

$$\hat{G} = \hat{J}\,\frac{1}{\tilde{G}^{-2} - \beta^2 \hat{J}^\dagger \hat{J}} , \tag{19}$$

and of the same type [cf. (8)]. We use the Fourier representation of $\hat{J}^\dagger\hat{J}$ and that of $\hat{G}$ and $\hat{D}$ to determine the ac response, $\partial\mathcal{M}(t)/\partial H(t')$, in terms of the single-site response functions ($L = Nl$):

$$\mathcal{M}(t) = \left(\frac{2e}{\hbar c}\right)^2 \left(\frac{L^2}{12}\right)^2 N \frac{J_0^2}{T}\,\frac{1}{\alpha^2}\left(1 - \frac{J_0^2}{A^2\alpha}\right) \times \int_{-\infty}^{t} G_{t-t'} D_{t-t'} [H(t) - H(t')]\, dt' . \tag{20}$$

We can insert the response and correlation functions found above to determine the ac susceptibility $\chi_\omega = \partial\mathcal{M}_\omega/\partial H_\omega$ which leads to

$$\chi_\omega = -\left(\frac{2e}{\hbar c}\right)^2 \frac{2N}{9}\left(\frac{L^2}{12}\right)^2 \frac{J_0\sqrt{\alpha}}{a}\,\frac{\omega}{\omega + 2i/\tau_R} , \tag{21}$$

where τ_R is the longest time scale of the response:

$$\tau_R \approx \begin{cases} \frac{2\tau_b}{a(\Theta)}, & \Theta > 0 , \\ \frac{2\mu\tau_b}{a_c}\left(\frac{\Theta_G}{\Theta - \Theta_G}\right)^\nu , & \Theta - \Theta \ll |\Theta_G| , \end{cases} \tag{22}$$

$$a(\Theta) = \Theta + \sqrt{\Theta^2 + 2\alpha} . \tag{23}$$

In Eq. (22) we see that the divergent relaxation time is directly observable in the physical ac magnetic response of the array. The $\omega \to 0$ limit of the ac susceptibility jumps to a finite value at $T = T_G$, indicating the development of a finite superconducting stiffness at the transition. Therefore measurement of this ac response in a fabricated array would probe its predicted glassiness.

In summary, we have presented a periodic model which displays "freezing" into one of an extensive number of metastable states without thermodynamic selection. Mode-coupling theory is exact for this long-range array, and its dynamical behavior approaching the onset of broken ergodicity is identical to that of the $p = 4$ (disordered) spherical model [9]. This glass transition at T_G is characterized by a diverging relaxation time and an accompanying jump in the Edwards-Anderson order parameter; the array's phase diagram is displayed in Fig. 1. It would be interesting to study the physical properties of this periodic model in its nonergodic regime ($T < T_G$); in particular, we expect "memory" effects in the form of an anomalous response function and "fingerprints" of the individual metastable states in its physical behavior. Since any uncertainty in the position of the wires introduces randomness in this array, it also offers the opportunity to study the crossover between glasses with spontaneously generated and quenched disorder.

We thank D. M. Kagan for alerting us to a mistake in the original text. M. V. F. acknowledges partial support from the International Science Foundation, the Russian Government (joint Grant M6M300), and the Russian Foundation for Fundamental Research (Grant No. 95-02-05720); M. V. F. and L. B. I. thank NECI for hospitality.

[1] See, e.g., T. R. Kirkpatrick and D. Thirumalai, Phys. Rev. B **36**, 5388 (1987); J. S. Yedidia and A. Georges, J. Phys. A **23**, 2165 (1990); L. F. Cugliandolo and J. Kurchan, Phys. Rev. Lett. **71**, 173 (1993); J. P. Bouchaud and M. Mezard, J. Phys. I (France) **4**, 1109 (1994); S. Franz and J. Hertz, Phys. Rev. Lett. **74**, 2115 (1995), and references therein.

[2] See, e.g. E. Marinari, G. Parisi, and F. Ritort, J. Phys. A **27**, 7647 (1994); M. Mezard and G. Parisi (unpublished).

[3] P. Chandra, L. B. Ioffe, and D. Sherrington, Phys. Rev. Lett. **75**, 713 (1995).

[4] C. de Dominicis and L. Peliti, Phys. Rev. B **18**, 353 (1978).

[5] H. Sompolinsky and A. Zippelius, Phys. Rev. B **25**, 6860 (1982).

[6] V. M. Vinokur, L. B. Ioffe, A. I. Larkin, and M. V. Feigelman, Sov. Phys. JETP **66**, 198 (1987).

[7] The high-temperature expansions for $\hat{D}(t)$ and $\hat{G}(t)$ in (12) differ from those of $D(t)$ and $G(t)$, respectively, by factors of $\beta^2 J^\dagger J$ which do not affect the latter's long-time behavior.

[8] W. Gotze, Z. Phys. B **56**, 139 (1984); E. Leutheusser, Phys. Rev. A **29**, 2765 (1984).

[9] A. Crisanti, H. Horner, and H.-J. Sommers, Z. Phys. B **92**, 257 (1993).