Scattering of Superfluid Vortex Rings

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The time evolution and scattering interactions of superfluid vortex rings are studied using the Gross-Pitaevskii model, in order to consider the maintenance and generation of vorticity in superfluid flows, and to compare ring dynamics in Navier-Stokes and superfluids. For a single ring, we verify previous analytic results on stability and the value of the translation velocity. When two vortex rings scatter, we generically observe merger and subsequent breakup in cases of close approach. Depending on the initial conditions, the final state can contain zero, two, or more final rings. [S0031-9007(96)00396-1]

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The dynamics of the turbulent state in both superfluids and normal Navier-Stokes (NS) fluids appears to depend crucially on the interactions between structures of localized vorticity–vortex filaments and rings. In the superfluid case [1], vorticity is associated entirely with such quantized filamentary structures, and calculations pioneered by Schwarz [2] have related the statistical properties of a persistent turbulent state to the dynamics of a "soup" of vortex filaments moving under the action of their own mutually induced velocity and that of the background normal fluid. In the Navier-Stokes case, experiments have identified significant vortex filamentary structure in turbulent flows, and numerous calculations have studied the corresponding flow evolution [3]. In either situation, the study of the flow in an entire region must be complemented with *local* calculations which consider the dynamics of two nearby segments of concentrated vorticity. In the NS case, several authors have considered the question of reconnection [4], but for superfluids rather less work has been done. In a previous Letter [5] we considered the reconnection of vortex filaments, showing that in the steady-state homogeneous flows considered by Schwarz [2], oppositely oriented filaments would always reconnect locally. Other related work has considered the origin and nucleation of vorticity due to interaction with walls [6].

In this Letter we consider the interaction of superfluid vortex *rings*. Our motivation is to understand the relation between vortex dynamics in normal and superfluids, both as a matter of scientific curiosity and as a practical device in computations. It is known [7] that the superfluid equations of motion reduce to the Euler equations of inviscid fluid dynamics outside the vortex cores, but the extent of analogy when core-scale dynamics applies is unclear. An extensive literature exists on NS vortex ring interactions (see [4,8] for surveys), inviting comparison with the superfluid case. Since the computational requirements of superfluid calculations are somewhat simpler than for NS, to the extent that the results are similar one may be able to obtain insight into ordinary turbulence with less effort.

As a microscopic description of superfluid dynamics we adopt the Gross-Pitaevskii model [9], a nonlinear Schrödinger equation for the boson wave function,

$$i\frac{\partial\psi}{\partial t} = -\nabla^2\psi + \psi[|\psi|^2 - 1]. \tag{1}$$

We have nondimensionalized the equation, and unit values of distance and time are about an angstrom and a picosecond, respectively. The superfluid density is $|\psi|^2$ and the flux is $2\,\mathrm{Im}(\psi^*\nabla\psi)$. A straight-line vortex filament is a solution of this equation of the form $\psi_1(\mathbf{x}) = f(r)e^{i\theta}$ in cylindrical coordinates. The function f has the limits $f \to 1$ as $r \to \infty$, corresponding to a vortical velocity field $\mathbf{v} = 2\hat{\theta}/r$, and $f \to 0$ as $r \to 0$, corresponding to a short distance regularization of the vortex core. An efficient algorithm for numerical integration of (1) in periodic boundary conditions is a split-step spectral Euler method [10] which we employed previously [5]; this method allows us to integrate in time preserving both the norm of ψ and the energy. In comparing to NS calculations, we should emphasize that there is no dissipation or vortex stretching present, and that the superfluid vortices have a compressible core [1]. Most calculations were performed on a CM-5, largely using a 128^3 Fourier decomposition, and we have verified the insensitivity of the results to changes in resolution or time step.

General vortex configurations are studied by evolving an appropriate initial condition. For one vortex ring, for example, at any point $\mathbf{x}$ we consider the plane passing through the point and containing the axis of the ring. The ring then intersects this plane at two points $\mathbf{x}_\pm$, once in the positive and once in the negative sense, and we take $\psi(\mathbf{x}, t=0) = \psi_1(\mathbf{x} - \mathbf{x}_+)\psi_1^*(\mathbf{x} - \mathbf{x}_-)$. A single vortex ring produced from this initialization translates indefinitely along its central axis due to its self-induced velocity field. The core oscillates slightly in time, but the ring is neutrally stable through many traversals of the box. Analytic calculations [11] have shown that a single vortex ring is linearly stable. Here, there is a slight subtlety due to the fact that the initial wave function is only approximate and does not have exactly the correct energy; since the numerical algorithm is energy conserving, this discrepancy persists in time and shows up as a harmless oscillation. The initialization generates rings whose core sizes are the same as that of a straight filament, and the stability of the resulting ring justifies this assertion. Lastly, we note that the value of the ring's velocity is in good agreement with the analytic result of Roberts and Grant [12].

Turning to ring-ring interactions, we first consider the mutual annihilation of two identical rings in an on-axis head-on collision, and in Fig. 1 we plot the surface $|\psi|^2 = 0.3$ at various times. (Other density values, or analogous surface plots of the energy density or the vorticity, yield similar pictures. Likewise, rings of different radii or other box sizes display the same behavior, provided the radius is large compared to the core size but small compared to the box size.) The initial value of ψ is just the product of the previous one-ring initial wave function and its complex conjugate (which then has reversed velocity), with the appropriate vertical separation. As the rings approach, their radii change only slightly until close approach, and the cores do not change at all until the separation is a few units. Contour plots of $|\psi|^2$ in a vertical plane bisecting the ring show a smooth merger of the respective isodensity curves, followed by their gradual disappearance as annihilation proceeds. The fact that substantial change in radius does not occur until the separation approaches the core size is familiar for normal fluid vortices, and occurs in both head-on collisions [13] and when a vortex nears a wall [14]. Provided the two cores do not overlap, this behavior can be understood from the classical reasoning in the latter paper, but of course the dynamics differ in the late stages. Two cases occur if two rings of *different* sizes approach on axis: if the ratio of radii is large, the rings simply leapfrog each other. If, however, the radii are comparable, one again sees annihilation similar to the equal-sized case, except that the partially overlapping rings continue to translate during the merger stage preceding annihilation.

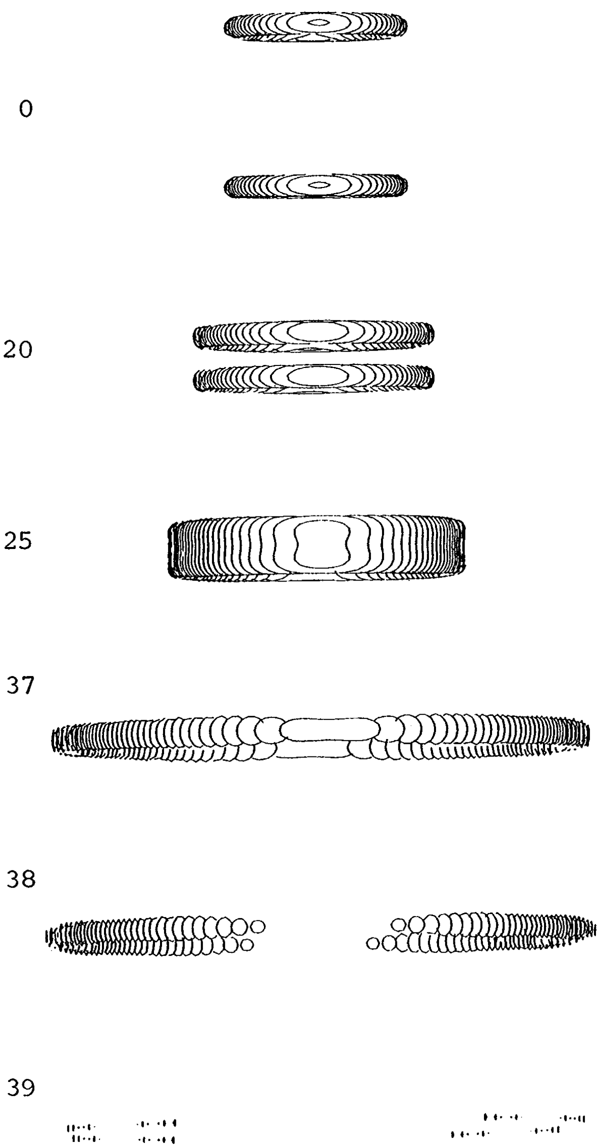

FIG. 1. Symmetric annihilation of two vortex rings; (dimensionless) times shown are 0, 20, 25, 37, 38, and 39.

The more general and interesting situation is the oblique interaction of vortex rings. In Fig. 2 we show a semisymmetric collision of two rings—the initial axes of the rings lie in a plane and intersect at 90°. In a manner similar to the NS case [15–17], the rings first contact and merge locally by the reconnection mechanism observed in [5], then the new ring bends up under its self-induced velocity, intersects itself, splits by the same reconnection mechanism, and then two new rings emerge in a plane orthogonal to the plane of incidence. Note that this is a scattering process in the standard quantum mechanical sense: initially there is a "free particle" state containing two separated circular rings moving essentially independently of each other, which upon close approach form a new and nontrivial "interacting" quantum state, which in turn decays to a different free particle state containing two new rings. The analogy to scattering was first made for vortex filament interactions in cosmological string models in 2D [18], and then for NS filaments in 3D [19], but only in the ring cases do the initial and final states correspond to *localized* particles.

We have simulated about twenty such collisions, varying the angle of intersection and the impact parameter, and to some extent the relative sizes of the rings. We have not seen an obvious pattern enabling us to predict the final state, but in general there is a change in the number and radius of emerging rings. For example, if the initial

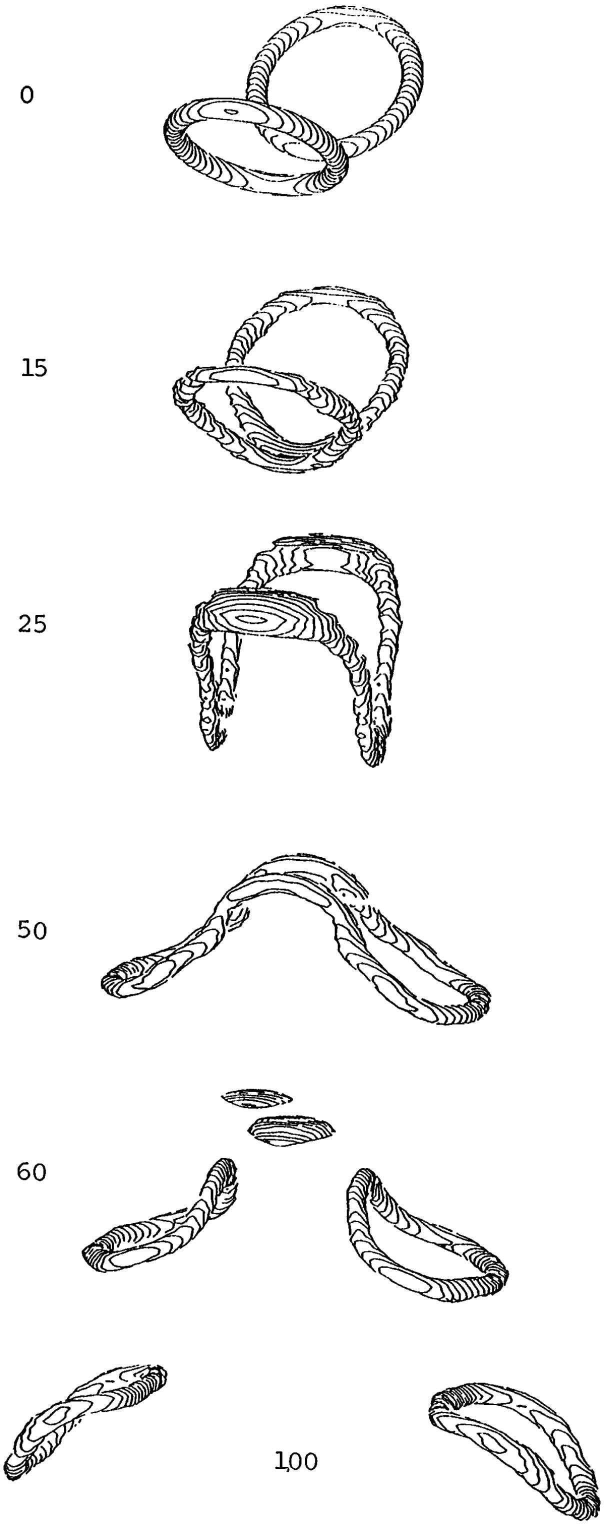

FIG. 2. In-plane collision of two vortex rings at 90° incidence, seen from the side. Times 0, 15, 25, 50, 60, and 100.

velocities intersect in plane at a slightly different angle, 120°, we obtain (Fig. 3) a four-ring final state. The final state contains more rings because the self-induced motion of the intermediate merged ring differs, and it happens to doubly intersect itself near its ends, rather than singly in

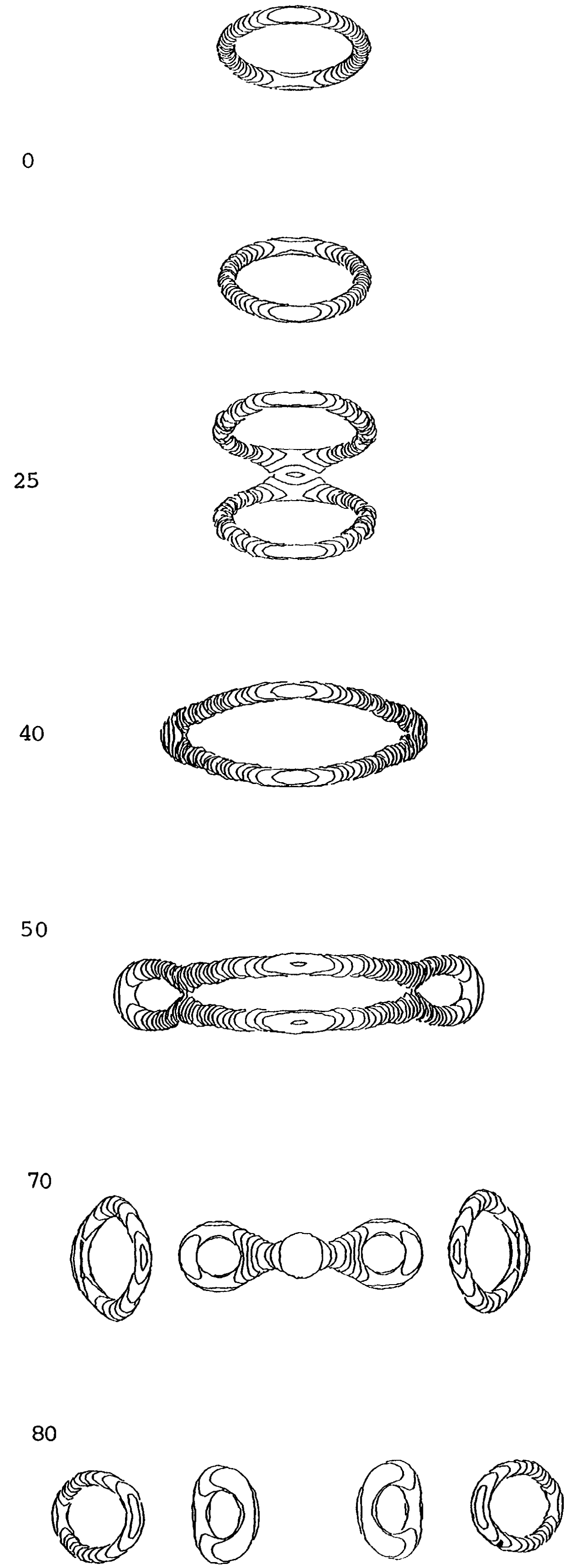

FIG. 3. In-plane collision of two vortex rings at 120° incidence, seen from above. Times 0, 25, 40, 50, 70, and 80.

its center as in the previous case. At later times, pairs of rings repeatedly collide, merge and break up. Off-center intersections have still different intermediate configurations, and may produce still different states. For example, a 90° intersection with a lateral displacement of one diameter gives three final rings. There is thus no evident restriction on the final state topology. When there *are* only two rings in the final state, they emerge roughly at 90° to the incident velocity plane as in Fig. 2, for small enough impact parameters. At larger values, for glancing collisions, the rings merge transiently in a local contact region but then split and go on their way with no ultimate deflection. If the rings pass nearby without direct contact, we see only a slight temporary shape perturbation.

These calculations are somewhat restricted in that they involve a small region of space with fixed energy and probability, which limits the possible final states. For example, while some collisions produce new ring structure, there is no net production of vorticity, in the sense that the total ring arclength has the same value in initial and final states. This behavior follows from the global conservations laws, combined with the requirement of a fixed core size. Realistically, density and energy fluctuations are possible as well, due to both thermal effects and radiation supplied from distant regions, so that still further vorticity configurations may arise. Thus, in a turbulent many-vortex state, multiple collisions with random initial orientations and impact parameters are likely to generate new vorticity. Finally, many of the vortex scattering results here resemble the NS results—the shape evolution in annihilation and the initial stages of oblique collision are the same, although the latter calculations have not reached the stage of isolated final rings. Note that while the previously established reconnection mechanism [5,19] makes the existence of multiring final states plausible, their detailed evolution requires multiple reconnections, some on length scales comparable to the core size, and the numerical calculations presented here are needed to determine which final states actually occur. The role of quantum mechanics in such processes is to limit possible core sizes and values of circulation to specific values, and in general to smooth the reconnection process (compare [5] to the NS analog in [19]). However, most of the ring evolution simply reflects the effects of the induced velocity, and this is the same in both cases. Superfluidz vortices may thus serve as a more tractable analog of vortex dynamics in normal turbulence.

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