Relative Particle Motion in Capillary Waves

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When a container of fluid is oscillated vertically, capillary waves develop on the surface if the amplitude exceeds a critical value. Experimentally one finds that the motion of small particles on the surface of the fluid is close to Brownian. Here we study the relative motion of particle pairs. The experiment establishes that particle motion is strongly correlated over macroscopic distances. Our observations are in striking agreement with upper-ocean studies, and with theories that appear applicable to this "weak turbulence" problem, and in disagreement with experimental and theoretical results for two-dimensional large-scale atmospheric turbulence. [S0031-9007(96)00448-6]

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As first pointed out by Richardson [1], the nature of turbulence is significantly illuminated by studying the motion of particle pairs, as opposed to the motion of a single fluid particle. He recognized that two particles separated by a distance R move more rapidly apart as R is increased. In particular, Richardson considered the relative diffusivity, here defined as $\langle d(R^2)/dt \rangle$ [2], where the angular brackets indicate an average over all particle pairs. He found that for atmospheric turbulence the relative diffusivity increases with separation as $R^{4/3}$. The Kolmogorov theory of three-dimensional fully developed turbulence, which came later [3], gave strong support to this observation. For two-dimensional fully developed turbulence, the relative diffusivity increases with separation as R^2 [4]. In accordance herewith, the relative diffusivity for atmospheric turbulence is found to have a crossover at $R \approx 20$ km, from the three-dimensional $R^{4/3}$ behavior to the two-dimensional R^2 behavior [5].

Fully developed turbulence is also referred to as "strong turbulence," to distinguish it from the type of turbulence called "weak turbulence" or "wave turbulence" [6]. While there are no small-amplitude homogeneous-background waves in strong turbulence, weak turbulence is characterized by the presence of small-amplitude waves having a dispersion relation $\omega(k)$. The experiments reported here probe turbulence of just this type. We track the motion of floating particle pairs sprinkled on a water surface in a cylindrical dish which is being oscillated vertically so as to create capillary waves [7]. When the amplitude A is just above the critical value A_c, where surface waves are first excited, there is no surface turbulence, and the capillary waves obey a dispersion relation $\omega^2(k) \propto k^3$. At stronger drive, nonlinear wave interactions are important [8], and the wave pattern becomes random in appearance. The theory of weak turbulence [6] takes these nonlinear interactions into account.

From our observations of relative particle motion we extract the relative diffusivity $d(R^2)/dt$. We analyze the distribution function $P\big(d(R^2)/dt\big)$ and its moments obtained at various values of R and various values of the reduced control parameter $\epsilon \equiv (A - A_c)/A_c$. We note that $\langle d(R^2)/dt \rangle = 2R\langle \delta v(R) \rangle$, where $\delta v(R) \equiv e_{\mathbf{R}} \cdot d\mathbf{R}/dt$ is the longitudinal component of the velocity difference, $e_{\mathbf{R}}$ being a unit vector in the direction $\mathbf{R}$. Thus, we have indirectly measured the distribution function for the velocity difference $\delta v(R)$ as well.

Our results are compared with the theory of weak turbulence, and with results found in upper-ocean studies [9–11]. The agreement is gratifyingly good. Our measurements, coupled with those of Ramshankar and Gollub [12], also enable us to test a prediction [13] which relates our measured R dependence of $\langle |\delta v(R)| \rangle$ to the fractal dimension d_f that characterizes the turbulence-induced contortion of a patch of dye floating on the water surface.

The oscillating dish had an interior diameter of 8.4 cm and a height of 2 cm, and the water filled the dish to a height of approximately 1 cm. The dish was mounted on a frame fixed to a Brüel and Kjaer vibration exciter type 4809, driven by a sinusoidal signal coming from a frequency synthesizer SRI model DS 345, operating at a frequency $f = 260$ Hz. At this frequency the wavelength was observed to be $\lambda = 2.6$ mm. The amplitude A of the vertical oscillations is proportional to the voltage applied to the exciter, and this applied voltage was the actual control parameter in the experiments. The particles used were mushroom spores. They were chosen because of their small mass and size (approximately 50 μm), and, more importantly, because they strongly resist being wetted by water, assuring that they will float. A charge-coupled device (CCD) camera recorded the images of the particles on a VCR tape, and the output of the VCR was fed into a frame grabber board in a personal computer.

For each value of ϵ (7 total), ranging from $\epsilon = 0.05$ to $\epsilon = 1.06$, approximately 1000 particle tracks were recorded. The tracking program arbitrarily selects 4 particles to track; for each particle the subsequent coordinate (x_i, y_i, t_i) is identified by searching a box with side length 2 mm, centered on the prior coordinates

$(x_{i-1}, y_{i-1}, t_{i-1})$ with $t_i - t_{i-1} = 20$ ms. A detailed discussion of the tracking program is found elsewhere [14]. First, we considered the self-diffusivity D_s [15]. By definition, $D_s \propto \overline{r^2(\tau)}/\tau$, where r is a distance a single particle moves in a time interval of length τ, and the bar denotes the average over initial times. In agreement with Ramshankar, Berlin, and Gollub (RBG) [15], we found that particles appear to move around quite randomly with a self-diffusivity that only increases slightly with time τ, $D_s \propto \tau^{2H}$, with values of the exponent $2H$ decreasing from 0.26 at $\epsilon = 0.05$ to 0 at $\epsilon = 1.06$. In comparison, RBG found the exponent $2H$ in the range 0.2 to 0 for $\epsilon = 0.06$ to 0.4.

Whereas the measurements of the self-diffusivity suggest that the particles are moving almost in a random-walk fashion, the relative diffusivity behaves very differently. For every particle pair with initial distance less than 4 mm, we recorded the separation $R(t)$ every 20 ms, resulting in more than 50 000 R values for each value of ϵ. Pairs separated by a distance smaller than or comparable to the pixel size ~0.3 mm were discarded by the tracking program. At large distances, our measurements were limited by the size of the camera window (circular with a radius of 3 cm ≈ 70% of the dish radius). The derivative $d(R^2)/dt$ for each particle pair at separation R is approximated by $\Delta(R^2)/\Delta t$—the change in R^2 over the fixed time $\Delta t = 20$ ms, divided by Δt. Over this period, the squared separation changes from R^2 to $R^2 + \Delta(R^2) = (\mathbf{R} + \delta\mathbf{v}\Delta t)^2$, where $\delta\mathbf{v}$ is the relative velocity. Thus,

$$\frac{\Delta(R^2)}{\Delta t} = \frac{(\mathbf{R} + \delta\mathbf{v}\Delta t)^2 - R^2}{\Delta t} = 2\mathbf{R} \cdot \delta\mathbf{v} + (\delta\mathbf{v})^2\Delta t\,. \tag{1}$$

It is necessary that Δt be small enough to assure that the term $(\delta\mathbf{v})^2\Delta t$ is negligible compared to the first term $2\mathbf{R} \cdot \delta\mathbf{v}$. Checks were made that this condition of small Δt is satisfied for the chosen value of $\Delta t = 20$ ms.

Figure 1 shows the relative diffusivity distribution $P(d(R^2)/dt)$ for $R = 3$, 6, and 14 mm, all at $\epsilon = 0.24$. The distributions are normalized, horizontally translated by their mean value $\langle d(R^2)/dt\rangle$, and rescaled by their standard deviation $\sigma(R) \equiv \{\langle[d(R^2)/dt]^2\rangle - \langle d(R^2)/dt\rangle^2\}^{1/2}$. The distributions consistently have their maximum to the left of the mean which is an order of magnitude smaller than $\sigma(R)$. For each value of R, the distribution includes distances between $R - \frac{1}{2}$ mm and $R + \frac{1}{2}$ mm. The size $N(R)$ of the ensemble of particle pairs at separation R depends on the distance R (see the inset of Fig. 1).

In Fig. 2, the mean value $\langle d(R^2)/dt\rangle$ as a function of R is shown on a log-log scale for seven values of ϵ ranging from 0.05 (lower line) to 1.06 (upper line). We find that the mean value increases with distance R. The error bars shown for $\epsilon = 1.06$ have a height equal to $\sigma(R)/\sqrt{N(R)}$, with similar error bars for the other ϵ values. For large values of R, the size of the ensemble is small (see the inset of Fig. 1), and the values obtained have large error bars. Assuming a power-law behavior $\langle d(R^2)/dt\rangle \propto R^\alpha$, we find $\alpha = 0.9 \pm 0.15$ [14], with a tendency to decrease with ϵ ($\alpha \simeq 1$ at small ϵ while $\alpha \simeq 0.8$ at larger ϵ). At $\epsilon = 1.06$ a slope is hardly defined. Using $\langle d(R^2)/dt\rangle = 2R\langle\delta v(R)\rangle$, we have $\langle\delta v(R)\rangle \propto R^\zeta$, where $\zeta = \alpha - 1$ is a small negative number that is possibly zero. We also extracted $\langle|d(R^2)/dt|\rangle = 2R\langle|\delta v(R)|\rangle \propto R^{\tilde{\zeta}+1}$ from our measurements. We find that $\tilde{\zeta}$ is very close to zero and slightly positive.

We examine our measurements in the light of a theory of Constantin and Procaccia (CP) [13]. Using a dye instead of particles, contours of constant dye concentration are described by their (homogeneous) fractal

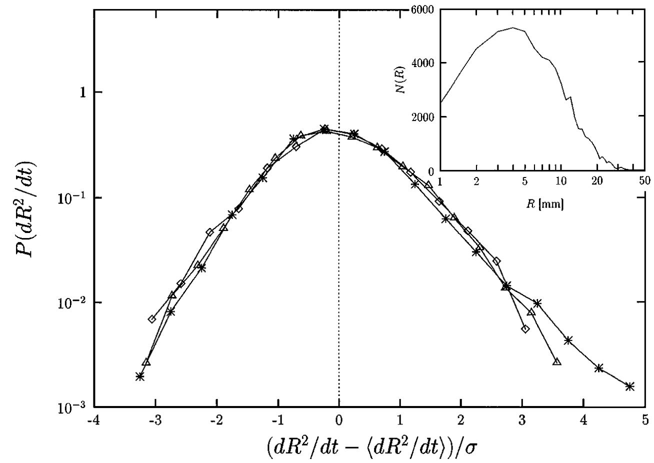

FIG. 1. Relative diffusivity distribution $P(d(R^2)/dt)$ (semi-logarithmic scale) for $R = 3$ mm (stars), $R = 6$ mm (triangles), and $R = 14$ mm (diamonds); $\epsilon = 0.24$. The error bars are of the size of the symbols. Inset: Sample size $N(R)$; $\epsilon = 0.24$.

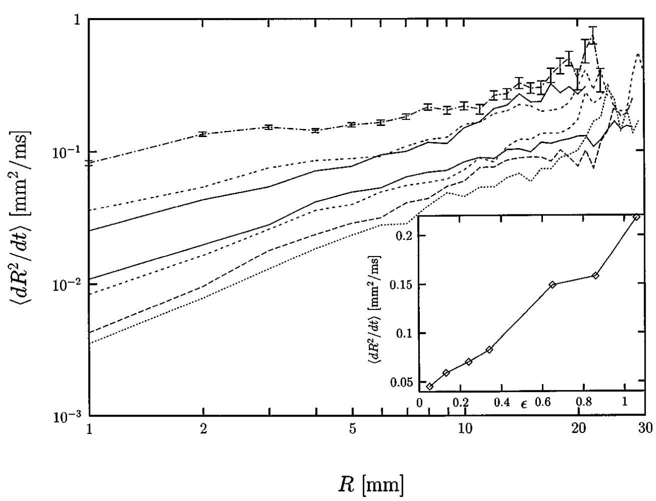

FIG. 2. Mean value $\langle d(R^2)/dt\rangle$ vs R (double-logarithmic scale). From below, $\epsilon = 0.05$, 0.13, 0.24, 0.34, 0.65, 0.86, and 1.06. Inset: Mean value $\langle d(R^2)/dt\rangle$ vs ϵ at $R = 10$ mm.

dimension d_f. In the CP theory, the dye concentration is a passive scalar; thus d_f is considered an intrinsic quantity related to the velocity field itself. According to the CP theory the exponents d_f, $\tilde{\zeta}$, and the dimension of the system d are related by $d_f = d - (1 - \tilde{\zeta})/2$. For capillary waves, the kinematic condition on the fluid surface [6], $\dot{\eta} = (-\partial_x \eta, -\partial_y \eta, 1) \cdot \mathbf{v}$, where $\eta(x, y)$ is the surface height and $\mathbf{v}$ is the incompressible velocity field, states that fluid particles at the surface only move along the surface. Thus, for surface-confined particles we can apply the CP relation with $d = 2$. Using the value $d_f = 1.40 \pm 0.05$ found for capillary waves in the dye experiment by Ramshankar and Gollub [12], the value $\tilde{\zeta} = -0.2 \pm 0.1$ is obtained. This result is in reasonable agreement with the value $\tilde{\zeta} \simeq 0$ which we find.

Next we examine the R dependence of the standard deviation, $\sigma(R)$, of $P(d(R^2)/dt)$ (Fig. 3). For each value of ϵ, the data are fitted rather well by a straight line, $\sigma \propto R^{\alpha_2/2}$, or, equivalently, $\langle[\delta v(R) - \langle\delta v(R)\rangle]^2\rangle \propto R^{\zeta_2}$, with $\zeta_2 = \alpha_2 - 2$ [see Eq. (1)]. Since the standard deviation $\sigma(R)$ is an order of magnitude larger than the mean $\langle d(R^2)/dt\rangle$, we also have $\langle\delta v(R)^2\rangle \propto R^{\zeta_2}$. For $\epsilon < 0.4$ we find $\zeta_2 = 0.26 \pm 0.06$. For larger values of ϵ, ζ_2 approaches zero. While the exponents ζ and ζ_2 vary only weakly with the control parameter, we find that the magnitude of the first and second moments of $P(d(R^2)/dt)$ increase roughly linearly with ϵ (see insets of Figs. 2 and 3).

We have also examined higher central moments, $\mu_n(R) = \langle[d(R^2)/dt - \langle d(R^2)/dt\rangle]^n\rangle$ and fitted the data by the algebraic form $\mu_n(R) \propto R^{n+\zeta_n}$. For $\epsilon = 0.24$ and $n = 3$, 4, and 6, ζ_n is well fit by $\zeta_3 = 0.37 \pm 0.06$, $\zeta_4 = 0.37 \pm 0.06$, and $\zeta_6 = 0.2 \pm 0.1$. Notice that ζ_n is not linearly proportional to n. The definitions of the various exponents are listed in Table I.

The capillary waves possess an underlying dispersion relation $\omega(k)$. Hence it seems appropriate to apply the theory of weak turbulence [6], which takes into account coupling of interacting modes in systems such as ours. The theory enables a calculation of the spectrum $\langle|\delta\mathbf{v}(k)|^n\rangle$ rather than the real space moments such as $\langle[\delta v(R)]^n\rangle$. In this theory, which neglects dissipation, $\langle|\delta\mathbf{v}(k)|^2\rangle$ for surface waves is found to be a product of two factors, an underlying velocity spectrum of $|\delta\mathbf{v}(k)|^2 \propto k^{5/2}$ and a weighting factor $n(k) \propto k^{-17/4}$ [6]. Keeping in mind that $\mathbf{k}$ is a two-dimensional vector, one shows, using the Wiener-Khintchine theorem [16], that

$$\langle|\delta\mathbf{v}(R)|^2\rangle \propto \int_{D(k_m)} |\delta\mathbf{v}(k)|^2 (1 - e^{2\pi i \mathbf{k}\cdot\mathbf{R}}) n(k)\, d^2k \propto 1 - a(R/\lambda)^{-1/4}, \quad (2)$$

where k_m is the maximum k value ($k_m \sim 1/\lambda$), and a is a constant of order unity (which may depend on ϵ). The integral is evaluated on a disk $D(k_m)$ of radius k_m. Thus, for large R, $\zeta_2 = 0$. For smaller R, the above relationship suggests a small positive value of ζ_2 [assuming $\langle\delta v(R)^2\rangle \propto \langle|\delta\mathbf{v}(R)|^2\rangle$], in agreement

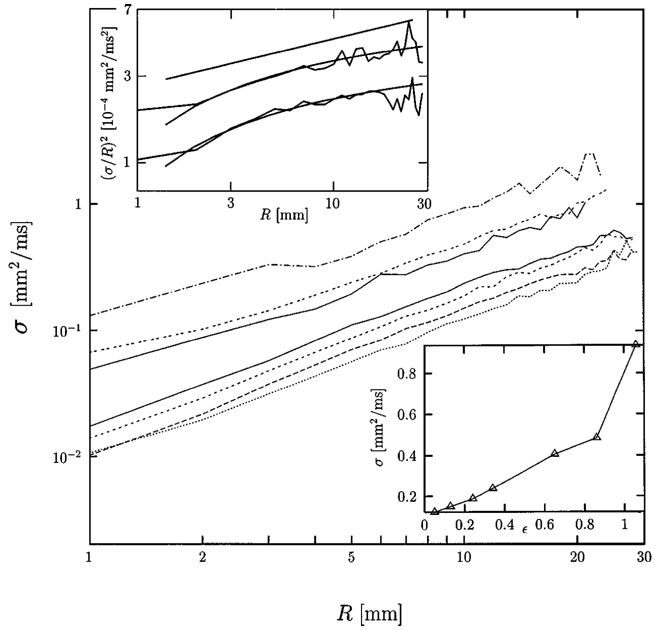

FIG. 3. Standard deviation σ vs R (double-logarithmic scale). From below, $\epsilon = 0.05$, 0.13, 0.24, 0.34, 0.65, 0.86, and 1.06. Lower inset: Standard deviation σ vs ϵ at $R = 10$ mm. Upper inset: $(\sigma/R)^2$ vs R on double-logarithmic scale for $\epsilon = 0.13$ (below) and $\epsilon = 0.24$ (above). Both sets of data are fit according to Eq. (2) with $a = 0.66$. The straight line has slope 0.26, our estimated value of ζ_2.

TABLE I. Definitions of exponents and a comparison between the exponents obtained from different experiments and theories.

$\langle d(R^2)/dt\rangle = 2R\langle\delta v(R)\rangle \propto R^{\alpha}$	
$\langle\delta v(R)\rangle \propto R^{\zeta}$, $\zeta = \alpha - 1$	
$\langle|d(R^2)/dt|\rangle = 2R\langle|\delta v(R)|\rangle \propto R^{\tilde{\zeta}+1}$	
$\sigma^2 \equiv \langle[d(R^2)/dt - \langle d(R^2)/dt\rangle]^2\rangle \propto R^{\alpha_2}$	
$\langle\delta v(R)^2\rangle \propto R^{\zeta_2}$, $\zeta_2 = \alpha_2 - 2$	
$\mu_n \equiv \langle[d(R^2)/dt - \langle d(R^2)/dt\rangle]^n\rangle \propto R^{n+\zeta_n}$	
This work, Faraday cell, mushroom spores	$\alpha = 0.9 \pm 0.15$ $\tilde{\zeta} = 0 - 0.1$ $\zeta_2 = 0.26$ ($\epsilon < 0.4$) $\zeta_2 = 0 - 0.2$ ($\epsilon > 0.4$)
Stommel [9], upper-ocean, floats	$\zeta_2 + 2 - \alpha = 4/3$
Okubo [10], upper-ocean, dye	$\alpha = 1.1$
Morel and Larcheveque [5], atmosphere, balloons	$\alpha = 2$, $\zeta_2 = 2$
Ramshankar and Gollub [12], Faraday cell, dye and CP relation [13]	$\tilde{\zeta} = -0.2 \pm 0.1$
Weak turbulence theory [6]	$\zeta_2 = 0$ (large R)

with our observations for $\epsilon < 0.4$ (recall that $\zeta_2 \simeq 0.26$). The theoretical fit (2) applied to our data for $(\sigma/R)^2 \propto \langle \delta v(R)^2 \rangle$ (upper inset of Fig. 3) shows that our observations are consistent with weak turbulence theory.

Our observations may also be compared with the upper-ocean studies [9–11] (Table I). In the work by Stommel [9], floats were released in pairs and followed over a large range of scales (10 cm to 1000 km, 3 s to 3 days). Over the various time scales Δt, the value of $\langle \delta v(R)^2 \rangle = \langle (\Delta R/\Delta t)^2 \rangle$ was found, and the relative diffusivity was defined as $\langle \delta v(R)^2 \rangle \Delta t$. From his results, Stommel found that $\langle \delta v(R)^2 \rangle \Delta t \propto R^{4/3}$. We can compare his results with ours, if we assume that $\langle R^2 \rangle \propto (\Delta t)^\beta$. Then, $d\langle R^2 \rangle/dt \propto (\Delta t)^{\beta-1} \propto R^{2-2/\beta}$, and $\langle \delta v(R)^2 \rangle \Delta t \propto R^{\zeta_2+2/\beta}$. Thus, $\alpha = 2 - 2/\beta$, and $\zeta_2 + 2/\beta = \zeta_2 + 2 - \alpha$. From Stommel's observations, we have $\zeta_2 + 2 - \alpha = 4/3$. This result is in good agreement with the value $\zeta_2 + 2 - \alpha = 1.25 \pm 0.1$ obtained from our values of α and ζ_2.

In a different study by Okubo [10], a dye is released. Okubo finds that the average distance of dye patches grows as $\langle R^2 \rangle \propto (\Delta t)^{2.3}$ (with Δt in the range a few hours to a month). As above, this implies [11] that the relative diffusivity $d\langle R^2 \rangle/dt \propto (\Delta t)^{1.3} \propto R^{1.1}$, compared to our result $\langle dR^2/dt \rangle \propto R^{0.9\pm0.15}$. However, the averaging process in Okubo's work is carried out for fixed time, not for fixed separation as in our case.

The results for the relative diffusivity obtained in our experiments and in upper-ocean studies are in sharp contrast to the results obtained by Morel and Larcheveque [5] for large-scale two-dimensional atmospheric turbulence. They measured the relative diffusivity of balloons distributed over the Southern Hemisphere. As we have done, they extract from their data both the mean value $\langle d(R^2)/dt \rangle$ and the width $\sigma(R)$ of $P\boldsymbol{(}d(R^2)/dt\boldsymbol{)}$, taking $\Delta t = 6000$ s. In accordance with Kolmogorov-like arguments applied to two-dimensional turbulence [4], they find $\alpha = 2$, $\zeta_2 = 2$, and $\langle R^2 \rangle \propto \exp[t/(1.35 \text{ days})]$.

In summary, measurements of the relative diffusivity $d(R^2)/dt$ establish that the surface motion of particles in our Faraday experiment is very far from being Brownian—in spite of contrary indications obtained from measurements of the single-particle diffusivity. For uncorrelated particles, $d(R^2)/dt$ is independent of R, whereas we find it to be roughly proportional to R. This strong R dependence establishes the existence of long-range velocity correlations, as in fully developed turbulence. However, the theory of fully developed turbulence, which applies to atmospheric turbulence, predicts results different from our results and from results obtained in upper-ocean studies. Instead, the theory of weak turbulence seems appropriate, and it predicts a scaling form for $\langle \delta v(R)^2 \rangle$ that is in excellent accordance with our measurements. Moreover, our findings seem roughly to support the Constantin-Procaccia relation between the exponent $\tilde{\zeta}$ for the absolute mean velocity difference and the fractal dimension d_f of isoconcentration contours formed by a passive scalar. The exponents α, $\tilde{\zeta}$, and ζ_2 obtained from the various experiments and theories are summarized in Table I.

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