Generalizing the Poynting Vector

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A very general energy conservation law derived from a Lagrangian theory of dielectric crystals is presented. It includes energy propagation from electromagnetic, spin, and acoustic waves. Both linear and nonlinear waves are included as well as various polaritonic combinations. Waves involving nonlocal (wave-vector–dispersive) interactions are also included. An example of the latter for which the Poynting vector is invalid, but which is correctly handled by this theory, is presented. [S0031-9007(96)00459-0]

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The Poynting theorem has been a cornerstone of electromagnetic theory since its publication in 1884. Its successes have been so many that its limitations have sometimes been forgotten or even denied. For example, a prominent textbook [1] argues for the general truth of the Poynting vector in the form $\mathbf{E} \times \mathbf{H}$ based on its normal component being continuous across a surface, a property that results when the tangential components of $\mathbf{E}$ and $\mathbf{H}$ are continuous across a surface. While the continuity of the normal component of the energy propagation vector across a surface is a necessary property, that property is not sufficient to guarantee the physical meaningfulness of a proposed energy propagation vector. Furthermore, recent work [2] on wave-vector– (i.e., spatially) dispersive media has shown that the continuity of tangential $\mathbf{H}$ (and also normal $\mathbf{D}$) across a surface can be violated owing to a surface layer intrinsic to such media.

The limitations of the Poynting theorem are apparent from a brief review of it as it applies to dielectric media. Maxwell's equations can be combined to yield

$$\mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0, \qquad (1)$$

where all quantities have their conventional meanings. This equation, however, is not in the form of a conservation law which must take the form of $\partial U/\partial t + \nabla \cdot \mathbf{S} = 0$. To obtain this form, linear constitutive relations,

$$\mathbf{D} = \epsilon_0 \overleftrightarrow{\kappa} \cdot \mathbf{E}, \qquad \mathbf{B} = \mu_0 \overleftrightarrow{\mu} \cdot \mathbf{H}, \qquad (2)$$

where $\overleftrightarrow{\kappa}$ and $\overleftrightarrow{\mu}$ are the relative dielectric and permeability tensors, are assumed. The well known result is

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E} \cdot \overleftrightarrow{\kappa} \cdot \mathbf{E} + \frac{\mu_0}{2}\mathbf{H} \cdot \overleftrightarrow{\mu} \cdot \mathbf{H}\right) + \nabla \cdot (\mathbf{E} \times \mathbf{H}) = 0. \qquad (3)$$

If the dependence on frequency ω of $\overleftrightarrow{\kappa}$ and $\overleftrightarrow{\mu}$ is considered [3], then the replacements $\overleftrightarrow{\kappa} \to d(\omega \overleftrightarrow{\kappa})/d\omega$ and $\overleftrightarrow{\mu} \to d(\omega \overleftrightarrow{\mu})/d\omega$ appear in Eq. (3).

What is shown, and only what is shown, by the Poynting theorem is that $\mathbf{E} \times \mathbf{H}$ is the energy propagation vector when $\mathbf{D}$ is related to $\mathbf{E}$, and $\mathbf{B}$ is related to $\mathbf{H}$ according to Eqs. (2). But all nonlinear relations between these four vectors, all wave-vector–dispersive interactions [those for which Eqs. (2) would involve space derivatives of one or more of the fields], and all interactions where magnetic effects are induced by electric variables and vice versa are not included. For these wide categories of interactions the Poynting theorem does not apply, and so $\mathbf{E} \times \mathbf{H}$ cannot be assumed to be the energy propagation vector and, in fact, is not. While it may come as no great surprise that the Poynting theorem does not apply to nonlinear interactions, a wave-vector–dispersive interaction, optical activity, had been known for over seventy years in 1884. For optical activity the $\mathbf{D}$ vs $\mathbf{E}$ relation of Eq. (2) must also contain a term involving the space derivative of $\mathbf{E}$ which leads to a linear dependence on the wave vector. Also, in recent decades linear propagation of light at frequencies near an exciton resonance in semiconductors has been of considerable interest. Such propagation involves resonant second-order wave-vector dispersion and so again the Poynting theorem does not apply, and $\mathbf{E} \times \mathbf{H}$ is inadequate for the energy propagation vector.

A fundamental tenet of the approach summarized here is that questions such as energy propagation in material media can be definitively answered only when the matter is treated on as fundamental a basis as the electromagnetic fields are, that is, with equations of motion for all long wavelength (continuum) material modes of excitation. Lax and the author were the first to couple Maxwell's equations to a deforming dielectric medium correctly in 1971 [4] as evidenced by the deduction from that theory that the photoelastic effect (the lowest order interaction of a deformation and the electromagnetic field) had been wrongly formulated for the 155 years of its existence [5]. The initial applications of this theory considered only electric dipole interactions and did not include wave-vector–dispersive interactions or magnetism arising from intrinsic spin. The formulation was published as a book [6] in 1979. Since then, magnetization and electric quadrupolarization effects from bound charge [2,7,8] have been included, wave-vector–dispersive interactions [2,9] have been explored, and magnetism from intrinsic spin has been incorporated with the use of Grassmann (anticommuting) variables [9–11]. The generality of the present formulation puts this theory in a unique position to discuss energy propagation in a crystalline dielectric medium.

The theory is Lagrangian based and begins from the point particle viewpoint with each particle endowed with a mass, a charge, and a Grassmann 3-vector which relates to spin. A long wavelength limit is performed in the Lagrangian to produce a continuum theory. The theory applies to crystalline media of arbitrary structural complexity and can be expanded straightforwardly to arbitrary nonlinear order in the interaction of, or between, the various modes of excitation—electromagnetic, acoustic, optic (both ionic and, to a certain extent, electronic), and spin. The electromagnetic equations in the Maxwell-Lorentz form (the form with only two electromagnetic fields **E** and **B**) are obtained from two parts of the Lagrangian density in the spatial system (position **z** and time t as variables), the field Lagrangian and interaction Lagrangian densities,

$$\mathcal{L}_S = \frac{1}{2}(\epsilon_0 \mathbf{E}^2 - \mathbf{B}^2/\mu_0) + \mathbf{j}\cdot\mathbf{A} - q\Phi . \qquad (4)$$

Here the electric and magnetic fields are related to the vector potential **A** and scalar potential Φ by $\mathbf{E} \equiv -\nabla\Phi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} \equiv \nabla\times\mathbf{A}$, q is the bound charge density, and **j** is the current density of bound charge and spin of the dielectric. When **A** and Φ are taken as the Lagrangian variables, the two Maxwell-Lorentz equations,

$$\frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mathbf{j}, \qquad \epsilon_0\nabla\cdot\mathbf{E} = q, \qquad (5)$$

result. The other two Maxwell equations are direct consequences of the definitions of the potentials.

An energy continuity equation can be formed from these equations in the form

$$\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) + \nabla\cdot\left(\frac{\mathbf{E}\times\mathbf{B}}{\mu_0}\right) = -\mathbf{j}\cdot\mathbf{E}. \qquad (6)$$

This equation states that the electromagnetic field subsystem has an energy sink that is a transfer of $\mathbf{j}\cdot\mathbf{E}$ energy to the matter subsystem. One can thus expect to find an energy continuity equation for the matter subsystem that has a source of just this magnitude.

The material equations of motion can be obtained from the interaction and matter Lagrangian densities in material coordinate **X** space, the latter given by

$$\mathcal{L}_{\mathcal{M}} = (\rho^0/2)(\dot{\mathbf{x}})^2 + (1/2)\sum_\nu m^\nu(\dot{\mathbf{y}}^{\mathbf{T}\nu})^2 + (i/2)\sum_\alpha \xi^{\mathbf{T}\alpha}\cdot\dot{\xi}^{\mathbf{T}\alpha} - \rho^0\Sigma, \qquad (7)$$

where **x** is the continuum center-of-mass position, $\mathbf{y}^{\mathbf{T}\nu}$ is a vector internal coordinate linearly related to the optic mode normal coordinates (T for total indicates it can have a spontaneous value in the unperturbed state), $\xi^{\mathbf{T}\alpha}$ is a Grassmann 3-vector related to spin by $\mathbf{s}^{\mathbf{T}\alpha} = -(i/2)\xi^{\mathbf{T}\alpha}\times\xi^{\mathbf{T}\alpha}$ (nonvanishing because of anticommutativity), Σ is the stored energy per unit mass, ρ^0 is the unperturbed mass density, and m^ν is the mass density associated with the ν internal coordinate. The dynamic term proportional to the imaginary unit i is not a kinetic energy as seen from its form and its disappearance from the energy density below. The stored energy Σ is a function of rotationally invariant measures (in the spatial system) of $\mathbf{s}^{\mathbf{T}\alpha}$, $\mathbf{y}^{\mathbf{T}\nu}$, and $\mathbf{x}_{,A}$, where $_{,A} \equiv \partial/\partial X_A$, and higher derivatives of these variables when the interaction considered requires them. Adopting definitions of these measures that vanish in the unperturbed state allows use of a series expansion in these measures to a level appropriate to the interaction.

The charge and current densities are expanded to electric quadrupole and magnetic dipole order as

$$q(\mathbf{z},t) = -\nabla\cdot\mathbf{P} + \nabla\nabla\colon\overleftrightarrow{Q}, \qquad (8)$$

$$\mathbf{j}(\mathbf{z},t) = \frac{\partial\mathbf{P}}{\partial t} + \nabla\times(\mathbf{P}\times\dot{\mathbf{x}}) - \frac{\partial}{\partial t}(\nabla\cdot\overleftrightarrow{Q}) - \nabla\times[\nabla\cdot(\overleftrightarrow{Q}\times\dot{\mathbf{x}})] + \nabla\times\mathbf{M}, \qquad (9)$$

$$\mathbf{P} \equiv J^{-1}\sum_\nu q^\nu\mathbf{y}^{\mathbf{T}\nu}, \qquad (10)$$

$$\overleftrightarrow{Q} \equiv (2J)^{-1}\sum_{\mu\nu} q^{\mu\nu}\mathbf{y}^{\mathbf{T}\mu}\mathbf{y}^{\mathbf{T}\nu}, \qquad (11)$$

$$\mathbf{M} \equiv \mathbf{M}^{\mathbf{c}} + \mathbf{M}^{\mathbf{s}}, \qquad (12)$$

$$\mathbf{M}^{\mathbf{c}} \equiv (2J)^{-1}\sum_{\mu\nu} q^{\mu\nu}\mathbf{y}^{\mathbf{T}\mu}\times\dot{\mathbf{y}}^{\mathbf{T}\nu}, \qquad (13)$$

$$\mathbf{M}^{\mathbf{s}} \equiv J^{-1}\sum_\alpha \mu^\alpha\mathbf{s}^{\mathbf{T}\alpha}. \qquad (14)$$

Here q^ν is a dipolar charge of the ν internal coordinate, $q^{\mu\nu}$ a quadrupolar charge, μ^α the magnetic moment of the α spin sublattice, and J is the Jacobian $|\partial\mathbf{x}/\partial\mathbf{X}|$.

Equations (5), (8), and (9) combine to produce the Maxwell equations provided **D** and **H** are defined by

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} - \nabla\cdot\overleftrightarrow{Q}, \qquad (15)$$

$$\mathbf{H} \equiv \mathbf{B}/\mu_0 - \mathbf{M} - \mathbf{P}\times\dot{\mathbf{x}} + \nabla\cdot(\overleftrightarrow{Q}\times\dot{\mathbf{x}}). \qquad (16)$$

The material equations of motion can now be obtained. The Lagrange equation for the center-of-mass continuum after considerable manipulation can be put in the form

$$\rho\ddot{x}_i = (q\mathbf{E} + \mathbf{j}\times\mathbf{B})_i + t^E_{il,l}, \qquad (17)$$

where $_{,i} \equiv \partial/\partial z_i$, $\rho \equiv \rho^0/J$, and the elastic stress tensor takes the form

$$t^E_{il} = \mathcal{E}_i P_l + \epsilon_{ijk}\left(\frac{\partial Q_{lj}}{\partial t} + (Q_{lj}\dot{x}_m)_{,m} + \epsilon_{ljm}M_m\right)B_k - 2\epsilon_{ijk}Q_{lm}\dot{x}_{j,m}B_k + 2Q_{lm}\mathcal{E}_{i,m} - (Q_{lm}\mathcal{E}_i)_{,m} + J^{-1}x_{l,A}\left(\frac{\partial\rho^0\Sigma}{\partial x_{i,A}} - \frac{d}{dX_B}\frac{\partial\rho^0\Sigma}{\partial x_{i,AB}}\right), \qquad (18)$$

where $\mathcal{E} \equiv \mathbf{E} + \dot{\mathbf{x}}\times\mathbf{B}$. The Lagrange equation for the internal coordinate $\mathbf{y}^{\mathbf{T}\nu}$, after some manipulation, becomes

$$m^\nu\ddot{y}^{T\nu}_i = q^\nu\mathcal{E}_i + \sum_\mu q^{\mu\nu}y^{T\mu}_j\mathcal{E}_{i,j} + \epsilon_{ijk}\sum_\mu q^{\mu\nu}\dot{y}^{T\mu}_j B_k - \epsilon_{ikl}\sum_\mu q^{\mu\nu}y^{T\mu}_j\dot{x}_{k,j}B_l - \frac{\partial\rho^0\Sigma}{\partial y^{T\nu}_i} + \frac{d}{dX_A}\frac{\partial\rho^0\Sigma}{\partial y^{T\nu}_{i,A}} - \frac{d^2}{dX_A dX_B}\frac{\partial\rho^0\Sigma}{\partial y^{T\nu}_{i,AB}}. \qquad (19)$$

The Lagrange equation for $\xi^{T\alpha}$ yields

$$\dot{\xi}_i^{T\alpha} = \epsilon_{ijk}\xi_j^{T\alpha}\left(\mu^\alpha B_k - \frac{\partial\rho^0\Sigma}{\partial s_k^{T\alpha}} + \frac{d}{dX_A}\frac{\partial\rho^0\Sigma}{\partial s_{k,A}^{T\alpha}}\right). \quad (20)$$

Interestingly, the spin $\mathbf{s}^{\mathbf{T}\alpha}$ can also be shown to obey Eq. (20) with $\xi^{\mathbf{T}\alpha}$ replaced by $\mathbf{s}^{\mathbf{T}\alpha}$.

An energy continuity equation of the matter subsystem can now be formed by adding the scalar product of $J^{-1}\dot{\mathbf{x}}$ with Eq. (17), the scalar product of $J^{-1}\dot{\mathbf{y}}^{\mathbf{T}\nu}$ with Eq. (19) (and define $\rho^\nu \equiv J^{-1}m^\nu$), and the scalar product of $-J^{-1}\mu^\alpha\mathbf{B}$ with Eq. (20) (with $\xi^{\mathbf{T}\alpha}$ replaced by $\mathbf{s}^{\mathbf{T}\alpha}$) and converting the material time derivatives of the inertial response terms to spatial frame time and space derivatives. After considerable rearrangements the energy continuity equation of matter has a source term that exactly cancels the sink term in Eq. (6), with the final result

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\rho}{2}\dot{\mathbf{x}}^2 + \sum_\nu \frac{\rho^\nu}{2}(\dot{\mathbf{y}}_i^{T\nu})^2 + \rho\Sigma - \mathbf{M}^{\mathbf{s}}\cdot\mathbf{B} + \frac{\epsilon_0}{2}\mathbf{E}^2 + \frac{1}{2\mu_0}\mathbf{B}^2\right) \\
&+ \frac{\partial}{\partial z_j}\Bigg[\left(\frac{\rho}{2}\dot{\mathbf{x}}^2 + \sum_\nu \frac{\rho^\nu}{2}(\dot{\mathbf{y}}_i^{T\nu})^2 + \rho\Sigma - \mathbf{M}^{\mathbf{s}}\cdot\mathbf{B}\right)\dot{x}_j - (t_{ij}^E + M_iB_j - \mathbf{M}\cdot\mathbf{B}\delta_{ij})\dot{x}_i \\
&+ [\mathbf{E}\times(\mathbf{B}/\mu_0 - \mathbf{M})]_j + \left(\dot{x}_{i,k}Q_{kj} - \frac{\partial Q_{ji}}{\partial t} - (Q_{ji}\dot{x}_k)_{,k}\right)\mathcal{E}_i \\
&- \frac{x_{j,A}}{J}\left(\frac{\partial\rho^0\Sigma}{\partial x_{i,BA}}\dot{x}_{i,B} + \sum_\nu \frac{\partial\rho^0\Sigma}{\partial y_{i,A}^{T\nu}}\dot{y}_i^{T\nu} + \sum_\nu \frac{\partial\rho^0\Sigma}{\partial y_{i,AB}^{T\nu}}\dot{y}_{i,B}^{T\nu} - \sum_\nu \dot{y}_i^{T\nu}\frac{d}{dX_B}\frac{\partial\rho^0\Sigma}{\partial y_{i,AB}^{T\nu}} + \sum_\alpha \frac{\partial\rho^0\Sigma}{\partial s_{i,A}^{T\alpha}}\dot{s}_i^{T\alpha}\right)\Bigg] = 0. \quad (21)
\end{aligned}$$

This is the most general statement of energy conservation in a dielectric yet obtained. It includes energy propagation from light waves, spin waves, optic mode waves somewhat away from Brillouin zone center, various polaritonic combinations of these excitations, nonlinear waves of each of these excitations, and wave-vector–dispersive modifications of these various waves. It also includes moving medium contributions to the energy propagation (for nonrelativistic matter velocities $\dot{\mathbf{x}}$).

The energy density contains six types of terms. In order, they are the kinetic energy of the center-of-mass continuum, the kinetic energy of the internal vibrations (optic modes), the stored energy of the bonding forces, a spin (only) magnetization energy, the electric field energy, and the magnetic field energy. Note the disappearance of the kinetic Lagrangian term involving the Grassmann variables; this shows that it is not a resident (kinetic) energy.

The energy propagation vector contains seventeen types of terms. Many represent moving medium effects ($\dot{\mathbf{x}} \neq 0$). The first four of those represent the transport of the parts of the energy density that are attached to the matter. The next three are the (total) magnetization-modified motion of stress. The next pair contain the first two terms in the definition of $\mathbf{H}$ in a vector product with $\mathbf{E}$. Note that the remaining terms in $\mathbf{H}$ do not match the multipole terms in the energy flow. Thus, the multipole terms cannot be gathered into an $\mathbf{E}\times\mathbf{H}$ Poynting vector term. In particular, note that among the following three quadrupole energy flow terms one term, $-\partial\mathbf{Q}/\partial t\cdot\mathbf{E}$, is not a moving medium term and is not a part of $\mathbf{E}\times\mathbf{H}$. It contributes importantly to linear optical propagation in optically active media (such as quartz) as recently shown [2]. The last five terms arise from wave-vector–dispersive interactions. The first is needed for acoustical activity and the second for optical activity. (In the absence of deformation, $x_{j,A}/J = \delta_{jA}$.) The next two terms have now been added to allow for second-order wave-vector dispersion in the internal coordinates. They permit demonstrating that a solution to the reflection and transmission problem near an exciton resonance conserves energy, a failing in previous work on this problem. The last term involving spin gradients is important in spin wave interactions.

In discussions of the Poynting vector the question of the uniqueness of a quantity appearing inside a divergence inevitably arises. It has been shown [6] that such a quantity is unique provided (i) the divergence term is in a conservation law (no sources or sinks), (ii) the normal component of the quantity is continuous across any surface (in particular, a body surface), fixed in the spatial frame, and (iii) the quantity reduces to the known form of that quantity in a vacuum. While these criteria work well at the electric dipole level of expansion, a problem arises at the electric quadrupole level because the term in the energy flow, $-(Q_{ji}\dot{x}_k)_{,k}\mathcal{E}_i$, has a spatial derivative of a material quantity, in this case $q^{\mu\nu}$, which becomes infinite over a zero distance when the standard "pillbox" argument is used with the Gauss theorem at a material surface. This is not surprising since it has been found [2] that the boundary conditions on the tangential components of $\mathbf{H}$ and the normal component of $\mathbf{D}$ fail when quadrupolarization terms are present for the same reason. This is a result of the inherent production of a surface layer, thin compared to a wavelength, by a wave-vector–dispersive (nonlocal) interaction to which the quadropolar interaction contributes. This problem has been solved by a new wave-vector space method [9] capable of solving problems of wave propagation in bounded media without using any boundary conditions. This method produces an alternative criterion to (ii) above, that is, that the normal component of the flow quantity, here the energy propagation vector, must be continuous across the surface layer, which being thin compared to a wavelength, arises naturally in the theory as surface distributions of the relevant fields.

Optical propagation at frequencies near the exciton resonance in crystals such as CdS has been a continuing problem of interest since the 1950s. Because the light wave mixes with the mobile polarizable exciton, two propagating polaritonic waves result, and it was initially believed that an *additional boundary condition (ABC)* was needed for the solution of the problem leading to it being termed the ABC problem. In the early 1970s a solution of the problem was obtained without an ABC by terminating the susceptibility by a step function at the crystal surface [12–14]. The interaction, however, is very nonlocal, involving second-order resonant wave-vector dispersion, and leads to a substantial surface layer effect. The severity of the abrupt termination assumption, called the dielectric approximation, was only later realized [15] when it was found to violate energy conservation at the surface in the transmission and reflection problem.

The wave-vector space method in conjunction with the energy conservation law, Eq. (21), resolves this energy problem completely. Being a macroscopic approach, it cannot, however, describe the nature of the surface layer, the object of much work in the field. Here an oversimplied model of the problem is presented, but it is one sufficient to illustrate the energy continuity at issue here. Let all first-order wave-vector dispersion terms arising from the stored energy and from the magnetization and quadrupolarization terms be dropped. The stored energy can be expanded as

$$\rho^0\Sigma = \sum_{\mu\nu} M_{ij}^{\mu\nu} y_i^\mu y_j^\nu + N_{ijkl} y_{i,j}^{\mathrm{ex}} y_{k,l}^{\mathrm{ex}} + O_{ijkl} y_i^{\mathrm{ex}} y_{j,kl}^{\mathrm{ex}}, \tag{22}$$

assuming for simplicity that the excitonic coordinate is uncoupled from the other internal coordinates. Cubic symmetry, $M_{ij}^{\mu\nu} = M\delta_{ij}\delta^{\mu\nu}$, $N_{ijkl} = N\delta_{ik}\delta_{jl}$, $O_{ijkl} = O\delta_{ij}\delta_{kl}$, is also assumed. The internal motion equations (19) are combined with the wave equation and the wave-vector space method is applied as in [9]. For simplicity, normal incidence from the vacuum is assumed. The transmission coefficients for the two polaritonic modes and the reflection coefficient are found to be

$$t_1 = 2(n_1^2 - n_{\mathrm{ex}}^2)/(n_1 - n_2)d, \tag{23}$$

$$t_2 = -2(n_2^2 - n_{\mathrm{ex}}^2)/(n_1 - n_2)d, \tag{24}$$

$$r = -(\kappa_b + n_1 n_2 - n_1 - n_2)/d, \tag{25}$$

$$d = \kappa_b + n_1 n_2 + n_1 + n_2, \tag{26}$$

where n_1 and n_2 are the refractive indices of the two polaritonic modes, κ_b is the background dielectric constant in the exciton resonance region, $n_{\mathrm{ex}} \equiv k_{\mathrm{ex}} c/\omega$, $k_{\mathrm{ex}}^2 \equiv k_\Omega^2(\omega^2 - \Omega^2)/\Omega^2$, $k_\Omega^2 \equiv M/(N - O)$, and $\Omega^2 \equiv 2M/m^{\mathrm{ex}}$. In terms of these quantities the wave-vector–dispersive exciton susceptibility is given by $\chi^{\mathrm{ex}}(k,\omega) = \chi_{\mathrm{ex}} k_\Omega^2/(k^2 - k_{\mathrm{ex}}^2)$ with $\chi_{\mathrm{ex}} \equiv (q^{\mathrm{ex}})^2/2\epsilon_0 M$.

Continuity of the time-averaged $\langle\,\rangle$ normal component of the energy propagation vector from Eq. (21) requires

$$\left\langle \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})_i \right\rangle_{\mathrm{norm}}^{\mathrm{vac}} = \left\langle \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})_i - \frac{\partial\rho^0\Sigma}{\partial y_{j,i}^{\mathrm{ex}}} y_j^{\mathrm{ex}} - \frac{\partial\rho^0\Sigma}{\partial y_{j,ki}^{\mathrm{ex}}} \dot{y}_{j,k}^{\mathrm{ex}} + \frac{d}{dX_k}\frac{\partial\rho^0\Sigma}{\partial y_{j,ki}^{\mathrm{ex}}} \dot{y}_j^{\mathrm{ex}} \right\rangle_{\mathrm{norm}}^{\mathrm{med}}. \tag{27}$$

Substitution of the fields into this equation leads to

$$1 - r^2 = n_1 t_1^2 + n_2 t_2^2 - \frac{n_1 t_1^2(n_2^2 - n_{\mathrm{ex}}^2)}{n_1^2 - n_{\mathrm{ex}}^2} - \frac{n_2 t_2^2(n_1^2 - n_{\mathrm{ex}}^2)}{n_2^2 - n_{\mathrm{ex}}^2}, \tag{28}$$

which is satisfied by Eqs. (23)–(26). Thus, the energy violation in previous work on this problem [15] is eliminated. This demonstrates the correctness and importance of the new energy flow vector.

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