First-Order Symmetric Hyperbolic Einstein Equations with Arbitrary Fixed Gauge

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(Received 26 January 1996)

We find a one-parameter family of variables which recast the 3 + 1 Einstein equations into first-order symmetric hyperbolic form for any fixed choice of gauge. Hyperbolicity considerations lead us to a redefinition of the lapse in terms of an arbitrary factor times a power of the determinant of the 3-metric; under certain assumptions, the exponent can be chosen arbitrarily, but positive, with no implication of gauge fixing. [S0031-9007(96)00413-9]

PACS numbers: 04.20.Ex

The issue of setting up a well-posed initial-value formulation for general relativity has been studied with the help of varied strategies, including special gauges and higher-order formulations [1]. Recently, a renewed interest [2] in the problem has arisen, in connection with the numerical evolution of the Einstein equations away from an initial hypersurface. Although the relevance of a manifestly hyperbolic formulation to the numerical integration of the Einstein equations is not yet clear, it is believed that a code tailored in a hyperbolic formulation would share properties of the exact system; namely, it would guarantee uniqueness and stability of solutions evolved from proper initial data. However, technical issues, associated with the discretization of the equations and the precision of the approximation, which may concern numerical stability, are not necessarily ruled out by a pure theoretical hyperbolic development.

In regards to the manifest hyperbolicity of the Einstein equations, the relevance of gauge choices has long been a question open to consideration. The gauge freedom of 3 + 1 general relativity is embodied by the lapse function and shift vector, which are completely arbitrary since their evolution is not determined by the theory. In general, a theory expressed in terms of equations on fields which admit gauge freedom may not admit a well-posed formulation unless improper gauge choices are ruled out, or the true gauge-invariant variables of the theory are found. A typical example is Maxwell's theory; on the one hand, it admits a hyperbolic formulation at fixed gauge and in terms of gauge-invariant variables as well; on the other hand, anomalous gauges can be found for which the resulting system does not have a well-posed initial-value formulation.

Our intention is to give an explicit argument to rewrite the 3 + 1 Einstein equations into a manifestly well-posed form, without the need of resorting to a choice of gauge. It has certainly been known that general relativity in special gauges can be set in symmetric hyperbolic form [3]. Furthermore, it has recently been shown that, for certain special first-order variables, general relativity admits a symmetric hyperbolic formulation for arbitrary fixed gauge [4,5]. Here we extend the existing results by showing that, under certain assumptions, there is a one-parameter family of new first-order variables for general relativity which satisfies first-order hyperbolic evolution for arbitrary but fixed choice of gauge.

Several concepts of hyperbolicity (e.g., strict, strong) can assert the well posedness of a system of partial differential equations (PDE's). Among all different concepts, symmetric hyperbolicity is especially appealing, for the reason that most interesting physical systems admit a formulation of this type [6]. Symmetric hyperbolicity is based on the symmetry properties of the differential operator [7]; therefore, multiple eigenvalues, which usually occur due to the presence of symmetries, play no role, as opposed to the case of other types of hyperbolicity. The reason for the well posedness in the symmetric-hyperbolic case is that the energy norm (an integral expression in terms of the fields) at later times can still be seen to be bounded by the norm at the initial time, because of cancellation of terms under integration by parts. The symmetry of the differential operator in the evolution equations guarantees the cancellation.

In the following, we set up the problem of general relativity in the 3 + 1 formulation due to Arnowitt, Deser, and Misner (ADM) in a noncanonical (though widely used) choice of variables, i.e., the intrinsic metric and the extrinsic curvature of the spatial hypersurfaces. We then redefine the variables in order to reduce the system to first order; the redefined variables depend on a set of parameters to be fixed by hyperbolicity considerations. Finally, we use the argument of the cancellation of terms under integration by parts in the energy norm to determine the parameters. In the process, we find that the lapse function must be redefined in terms of the determinant of the 3–metric, without any loss of gauge freedom.

In order to fix notation, we summarize some necessary points of the 3 + 1 formulation. The 3 + 1 splitting of the fully four-dimensional formalism consists of a spacelike foliation by the level surfaces of a function

$t(x^a)$. The unit normal form is $n_a = -N\nabla_a t$, where N is the lapse function. The unit normal vector is given by $n^a = \frac{1}{N}(t^a - N^a)$, where N^a is the shift vector. The metric g^{ab} induces a 3-metric on the spatial surfaces, by $h^{ab} = g^{ab} + n^a n^b$. In a coordinate system $\{x^0, x^i\}$, $i = 1, 2, 3$, adapted to the surfaces (such that $t = x^0$), the induced metric h^{ab} reduces to h^{ij}. The extrinsic curvature of the 3-surfaces is defined by $K^{ab} \equiv \frac{1}{2}\pounds_n h^{ab}$, and is also a 3-tensor.

The equations for the evolution of the intrinsic contravariant metric h^{ij} can be taken as (from [8], with the notation of Chap. 10 and App. E of [9])

$$\dot{h}^{ij} = 2NK^{ij} - D^i N^j - D^j N^i, \tag{1}$$

$$\begin{aligned}\dot{K}^{ij} = N[R^{ij} - 2K^{ik}K^j{}_k + K^{ij}K \\ - \kappa(S^{ij} - \tfrac{1}{2}h^{ij}(S - \rho))] - D^i D^j N \\ + N^k D_k K^{ij} - K^{ik} D_k N^j - K^{jk} D_k N^i.\end{aligned} \tag{2}$$

The notation $(\dot{\cdot})$ stands for $\pounds_{t^a}$ or simply $\partial/\partial t$. Indices $i, j, k, \ldots$ are raised and lowered with the 3-metric h^{ij}; the operator D_i is the covariant derivative with respect to the 3-metric h_{ij} [9]. For any 3-tensor U^{ij}, the notation U stands for its trace with respect to the 3-metric: $U \equiv U^k{}_k$. The matter tensor S^{ij} is the projection of the four-dimensional stress-energy tensor T^{ab} into the spatial hypersurfaces, and ρ is the projection of T^{ab} in the direction normal to the surfaces. The results derived in this Letter do not depend strongly on the particular matter source, but hold for any sources that admit a first-order symmetric hyperbolic formulation on their own.

Equations (1) and (2) for the fields (h^{ij}, K^{ij}) are supplemented by the constraints

$$C := \tfrac{1}{2}(R + K^2 - K_{ij}K^{ij}) - \kappa\rho = 0, \tag{3}$$

$$C^i := D_j K^{ij} - D^i K - \kappa J^i = 0, \tag{4}$$

where J^i is the mixed projection of T^{ab} onto the hypersurface and the normal. If C and C^i can be shown to be conserved as a consequence of (1) and (2), then the constraints need only to be imposed on an initial hypersurface. This will be our point of view in the following.

Introduction of the parameters.—In order to show that there exists a one-parameter family of variables for which general relativity takes a first-order symmetric hyperbolic form, we first introduce a set of four parameters, α, β, γ, and ϵ; we eventually require the parameters to satisfy a set of three algebraic conditions that guarantee the hyperbolicity.

Two parameters, α and β, are used to redefine variables as follows:

$$M^{ij}{}_k \equiv \tfrac{1}{2}(h^{ij}{}_{,k} + \alpha h^{ij} h_{rs} h^{rs}{}_{,k}), \tag{5}$$

$$P^{ij} \equiv K^{ij} + \beta h^{ij} K. \tag{6}$$

The definition (5) reduces the Einstein equations to first order. Note that the variable $M^{ij}{}_k$ represents the spatial derivative of the densitized 3-metric, $M^{ij}{}_k = \frac{1}{2}h^\alpha(h^{-\alpha}h^{ij})_{,k}$, where h is the determinant of h_{ij}. Equations (5) and (6) can be inverted into

$$h^{ij}{}_{,k} \equiv 2\left(M^{ij}{}_k - \frac{\alpha}{3\alpha + 1} h^{ij} M_k\right), \tag{7}$$

$$K^{ij} \equiv P^{ij} - \frac{\beta}{3\beta + 1} h^{ij} P, \tag{8}$$

with the notation $M_k \equiv h_{ij} M^{ij}{}_k$.

A third parameter, γ, is introduced in the evolution equations to allow for a combination of (1) and (2) with the constraints (3) and (4). In this way, the principal part of the evolution equations (1) and (2) can be modified. The constraints (3) and (4) will be assumed to be conserved by the resulting equations. Since γ plays a crucial role in the hyperbolicity of the system, in the following we point out its exact place in the evolution equations.

The evolution equation for $M^{ij}{}_k$ can be obtained by, first, taking a space derivative $\partial/\partial x^k$ of Eq. (1), and then tracing and combining the resultant equation according to the definition (5). We also add the vector constraint C^i with an appropriate—uniquely determined—factor, obtaining the following:

$$\begin{aligned}\dot{M}^{ij}{}_k = \tfrac{1}{2}((\dot{h}^{ij})_{,k} + \alpha(\dot{h}^{ij} h_{rs} + h^{ij}\dot{h}_{rs})h^{rs}{}_{,k} \\ + \alpha h^{ij} h_{rs}(\dot{h}_{rs})_{,k}) - N\delta_k^{(i} C^{j)}.\end{aligned} \tag{9}$$

If Eq. (1) is used in the right-hand side of (9) to eliminate time derivatives in favor of space derivatives of the new fields, the right-hand side becomes a combination of the fields h^{ij}, $M^{ij}{}_k$, P^{ij}, lapse, shift, and sources; first-space derivatives of $M^{ij}{}_k$, P^{ij}, lapse, and shift; and second-space derivatives of the shift. This equation is shown explicitly below [Eq. (17)], correct to principal terms.

The evolution equation for P^{ij} is obtained directly from Eq. (2), by the appropriate combination with its trace, as prescribed by the definition (6). We also add the scalar constraint C with a suitable factor:

$$\begin{aligned}\dot{P}^{ij} = \dot{K}^{ij} + \beta(\dot{h}^{ij}K + h^{ij}\dot{h}_{rs}K^{rs} + h^{ij}h_{rs}\dot{K}^{rs}) \\ + 2N\gamma h^{ij} C.\end{aligned} \tag{10}$$

When Eqs. (1) and (2) are substituted appropriately in (10), the right-hand side becomes a combination of the fields h^{ij}, $M^{ij}{}_k$, P^{ij}, lapse, shift, and sources; first-space derivatives of $M^{ij}{}_k$, P^{ij}, lapse, and shift; and second-space derivatives of lapse. This equation is shown explicitly below [Eq. (16)], correct to principal terms.

The fourth parameter, ϵ, is introduced in order to redefine the lapse N by

$$N \equiv h^{-(3\alpha+1)\epsilon/2} Q, \tag{11}$$

for an arbitrary function Q. The lapse is thus redefined without loss of generality; the gauge freedom is transferred to Q, and the parameter ϵ remains to be specified. Notice that

$$\frac{N_{,k}}{N} \equiv \epsilon M_k + (\ln Q)_{,k}. \tag{12}$$

Since second derivatives of the lapse appear in (10), this redefinition allows for a modification of the principal terms in (10).

Hyperbolicity imposed on the system.—We define the "energy norm" of the system at time t as

$$E(t) = \frac{1}{2}\int_\Sigma h^{ij}h_{ij} + P^{ij}P_{ij} + M^{ij}{}_k M_{ij}{}^k, \tag{13}$$

where the integration is performed on the surface Σ defined by $t = \text{const}$. The spatial symmetry of the system is guaranteed if the principal terms in the time derivative of the energy [10] can be combined into total divergences, since in this case their contribution to the energy estimates would vanish.

The time derivative of the energy, correct to principal terms, is

$$\dot{E}(t) = \int_\Sigma \dot{h}^{ij}h_{ij} + \dot{P}^{ij}P_{ij} + \dot{M}^{ij}{}_k M_{ij}{}^k. \tag{14}$$

The evolution equations (1), (9), and (10) can be used to trade time derivatives for space derivatives in (14). If the principal terms can be eliminated under integration by parts, then the system becomes hyperbolic.

In the following we write the principal terms of the evolution equations and find the conditions that are necessary to symmetrize the system.

The Ricci tensor R_{ij} is needed in terms of the new fields. Recall

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ki,j} + \Gamma^k_{ij}\Gamma^l_{kl} - \Gamma^k_{lj}\Gamma^l_{ki},$$

with $\Gamma^k_{ij} = -\frac{1}{2}h_{il}h^{kl}{}_{,j} - \frac{1}{2}h_{jl}h^{kl}{}_{,i} + \frac{1}{2}h^{kl}h_{ir}h_{js}h^{rs}{}_{,l}$. In terms of $M^{ij}{}_k$, the connection Γ^k_{ij} takes the form

$$\Gamma^k_{ij} = -2h_{l(i}M^{kl}{}_{j)} + h^{kl}h_{ir}h_{js}M^{rs}{}_l + \frac{2\alpha}{3\alpha+1}\delta^k_{(i}M_{j)} - \frac{\alpha}{3\alpha+1}h_{ij}h^{kl}M_l. \tag{15}$$

The principal part of (10) is then

$$\dot{P}^{ij} = N^k P^{ij}{}_{,k} + N\left(h^{kl}M^{ij}{}_{k,l} - 2h^{l(i}M^{j)k}{}_{l,k} + \frac{2\alpha+1}{3\alpha+1}h^{ik}h^{jl}M_{k,l} + \frac{2\beta(\alpha+1)-\alpha}{3\alpha+1}h^{ij}h^{kl}M_{k,l} - 2\beta h^{ij}M^{kl}{}_{k,l}\right) - h^{ik}h^{jl}N_{,kl} - \beta h^{ij}h^{kl}N_{,kl} + 2N\gamma h^{ij}\left(-M^{kl}{}_{k,l} + \frac{\alpha+1}{3\alpha+1}h^{kl}M_{k,l}\right). \tag{16}$$

The principal terms of (9) are the following:

$$\dot{M}^{ij}{}_k = N^l M^{ij}{}_{k,l} + N\left(P^{ij}{}_{,k} + \frac{\alpha-\beta}{3\beta+1}h^{ij}P_{,k}\right) - 2N\delta^{(i}_k P^{j)l}{}_{,l} + 2N\frac{\beta+1}{3\beta+1}\delta^{(i}_k h^{j)l}P_{,l}. \tag{17}$$

In view of (16) and (17), the cancellation under integration by parts in (14) takes place if the following algebraic conditions are imposed on the parameters α, β, γ, and ϵ:

$$\frac{2\alpha+1}{3\alpha+1} - \epsilon = 0, \tag{18a}$$

$$\frac{\beta+1}{3\beta+1} + \beta + \gamma = 0, \tag{18b}$$

$$\frac{2(\beta+\gamma)(\alpha+1)-\alpha}{3\alpha+1} - \frac{\alpha-\beta}{3\beta+1} - \beta\epsilon = 0. \tag{18c}$$

Condition (18a) has the effect of the cancellation of the fourth and seventh terms in (16), even before their contribution to the energy is considered. This is done in this way, because the fourth term in (16) has no symmetric counterpart in (17) with respect to its contribution to the energy, and, therefore, needs to be eliminated from the system. Condition (18b) guarantees the cancellation, under integration by parts, of the sixth and ninth terms in (16), together with their symmetric counterpart, i.e., the fifth term in (17). Lastly, condition (18c) guarantees the symmetry of the fifth, eighth, and tenth terms in (16) with the third term in (17), which subsequently make no contribution to $\dot{E}$.

With the assumptions that $Q > 0$, that h^{ij} is positive definite, that the algebraic conditions (18) are met by the four parameters α, β, γ, and ϵ, and that the constraints C and C^i are conserved, the fields $(h^{ij}, M^{ij}{}_k, P^{ij})$ satisfy a symmetric hyperbolic system of PDE's, namely Eqs. (1), (9), and (10), with the initial data constrained by (3), (4), and (5).

Notice that the conditions (18) leave free one of the four parameters. Any one of the parameters can be chosen freely, within a real range that allows for real values for the remaining three parameters as solutions of (18). For instance, if α is considered as the free parameter, then α can take values in $(-\infty, -1/2)$, while β must be chosen as a root of the following quadratic equation:

$$3\beta^2 + 2\beta + \frac{(3\alpha+1)(\alpha+1)+1}{(2\alpha+1)} = 0. \tag{19}$$

A most interesting choice of variables is $P^{ij} = K^{ij} - h^{ij}K$ (proportional to the canonical ADM momentum), or $\beta = -1$. This choice of β, in turn, fixes $\gamma = 1$, $\alpha = -1$, and $\epsilon = 1/2$. The redefinition of lapse becomes $N = Q\sqrt{h}$, and the number of terms in the principal parts in Eqs. (16) and (17) reduces considerably. Regarding the propagation of the constraints, for $\gamma = 1$ it can be shown that the Bianchi equations imply a homogeneous symmetric hyperbolic evolution system for C and C^i. It follows that the constraints are conserved. This case was explored earlier by the authors, and has been found suitable for the development of a smooth Newtonian limit [4] if certain gauge choices are imposed in addition to the well-posed formulation. Most remarkably, for $\gamma \neq 1$ the evolution of the constraints is not symmetric hyperbolic nor strictly hyperbolic, and the validity of the assumption of the conservation of the constraints must be studied carefully. The details will soon appear elsewhere.

Equation (18a) shows that the exponent of $\sqrt{h}$ in the redefinition of the lapse (11) is equal to $-(2\alpha + 1)$, being thus any positive real number, but never zero. Thus, it is not possible to have a set of variables of the form (5) and (6) with symmetric hyperbolic evolution without relating the true lapse N to the 3-metric.

The system (1), (9), and (10) has a nontrivial set of characteristics [11]. Using the notation $\xi_a \equiv (v, \xi_i)$ where ξ_i has unit norm with respect to h^{ij}, it is immediate to see that ξ_a is characteristic if $t^a \xi_a = 0$ (for any α). Furthermore, by essentially the same arguments as in [5], it can be shown that covectors ξ_a satisfying either $\xi^a \xi_a = 0$ or $n^a \xi_a = 0$ are also characteristic (for any α). There are no other characteristics if α takes the value -1, as in [4]. Therefore, if $\alpha = -1$, the characteristics are null (with speeds $N^i \xi_i \pm N$), or timelike and either tangent to t^a (with zero speed) or tangent to the normal direction n^a (with speed $N^i \xi_i$). However, if $\alpha \neq -1$ the system may have other characteristics, *in addition to these*, with speeds that may depend (nontrivially) on the choice of α. The details will also appear elsewhere.

In this work the choice of N^i and Q is *arbitrary* but *given.* The gauge must be specified in order to integrate the equations. On the other hand, the hyperbolicity of the system holds independently of the choice of gauge. The fact that gauge fixing is required is not troublesome; the variables themselves are not gauge-invariant fields.

In order to avoid confusion, we point out that by fixing a gauge we understand a nondynamical specification of N^i and Q as *a priori* known functions for all time, independent of the evolution of the new fields. In this way, N^i and Q act as known sources. If the gauge were specified dynamically (explicitly, or implicitly via an equation) as a function of the new fields, then the principal part of the evolution equations would be modified. The results proven here do not guarantee the well-posed evolution of such a choice of gauge; in fact, it is not hard to see that, in general, the well posedness would be hampered.

The system shown here shares, with other hyperbolic formulations, the property of manifest first-order flux-conservative form, which makes it suitable for the application of general numerical integration techniques [12] (no numerical applications of this formalism have been investigated as of now). Aside from that, we find it very appealing for its remarkable simplicity and clarity.

It is our pleasure to thank H. Friedrich for valuable suggestions and criticisms. O. A. Reula acknowledges support from CONICOR and SECYT (Argentina). S. Frittelli acknowledges partial support from NSF.

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