Generalized Nonextensive Thermodynamics Applied to the Cosmic Background Radiation in a Robertson-Walker Universe

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Statistical mechanics is useful to introduce generalizations of standard thermodynamics through the generalization of the entropy and other state functions. Along these lines the Tsallis nonextensive and the Bergmann group symmetric generalizations have proven to be very useful. We combine both formalisms to describe the nonextensive thermostatistics in a relativistic setting. We obtain the generalized forms of the first and second laws of thermodynamics for reversible processes, and apply the resulting theory to the cosmic blackbody radiation in a Robertson-Walker model of the Universe. We show that the temperature of the cosmic blackbody radiation varies as the inverse of the scale factor of the Universe, and is independent of the degree of nonextensivity. [S0031-9007(96)00423-1]

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The thermodynamics of a general system is contained in the knowledge of two state functions of its thermodynamics parameters. Usually, we are satisfied with the internal energy and the entropy of the system. The other state functions, which are useful in determining the equilibrium state of a system that is not isolated, are obtained as Legendre transformations involving those initial functions.

Many methods have been used to derive the internal energy and entropy of a system. Among them we find experimental and theoretical procedures, and within this last group we can mention kinetic theory or, more generally speaking, statistical mechanics and information theory. In particular, the statistical mechanics approach is very useful when we wish to introduce generalizations of the standard thermodynamics (thermostatistics) through the generalization of the entropy and other basic state functions. Along these lines the group symmetric [1–3] and the nonextensive generalizations [4] have proven to be useful. Indeed, on one hand, the group symmetric approach has provided an interesting insight into the formulation of relativistic thermodynamics and its application to systems for which a unique rest frame does not exist [3]. On the other hand, the nonextensive statistical approach has been successful in the context of the Lévy-like anomalous diffusion [5], of the solar neutrino problem [6], of electron plasma two-dimensional turbulence [7], and of optimization techniques [8].

In this Letter we wish to show that it is possible to combine both generalizations to obtain a simple and elegant formalism, including the generalized forms of the first and second laws of thermodynamics, which may be applied, for instance, to describe the nonextensive thermodynamics of the cosmic blackbody radiation [9,10], starting from the description of a single pencil of radiation for which a *unique rest frame does not exist.* We treat this particular example, for a class of cosmological models of the Universe, at the end of our Letter.

Let $\rho \equiv (\rho_1, \rho_2, \ldots)$, with $\rho_s \geq 0$ and $\sum \rho_s = 1$, be a discrete probability distribution characterizing the state of a physical system (in a quantum mechanical version ρ is the density operator). From now on we shall use the density operator formalism for simplicity in the notation, and take the Boltzmann constant equal to 1. Consider the q-entropy functional [11]

$$S_q[\rho] = [1 - \mathrm{tr}(\rho^q)](q - 1)^{-1}, \tag{1}$$

where q is a positive real number. We verify that $\lim_{q\to 1} S_q[\rho] = -\mathrm{tr}(\rho \ln \rho)$.

We shall say that a system with given q-expectation values U_q^j (j may represent a set of tensorial indices) for a set of observables U^j is in thermodynamic equilibrium if its density operator maximizes the value of the q-entropy functional under the subsidiary conditions

$$\mathrm{tr}(\rho) = 1, \quad \mathrm{tr}(\rho^q U^j) = U_q^j. \tag{2}$$

Therefore, by using the method of Lagrange multipliers, we find that ρ is an extremal for the functional $S_q[\rho]$, consistent with conditions (2) if and only if

$$\left(\frac{q}{1 - q} + q \sum_j \beta_j U_s^j\right)\rho_s^{q-1} + \gamma = 0, \tag{3}$$

where β_j and γ are Lagrange multipliers. Equation (3) gives

$$\rho_s = Z_q^{-1}\left[1 - (1 - q)\sum_j \beta_j U_s^j\right]^{1/(1-q)}. \tag{4}$$

The q-partition function, Z_q, is given by the expression

$$Z_q(\beta_j; \lambda) = \mathrm{tr}\left[1 - (1 - q)\sum_j \beta_j U^j\right]^{1/(1-q)}, \tag{5}$$

with λ representing a possible set of external parameters entering the definition of the operators U^j. In the $q \to 1$

limit we recover the standard expression of the partition function of a canonical ensemble.

To satisfy the conditions (2) we have to use (5) and formally compute the values of β_j from

$$U_q^j = -\frac{\partial}{\partial \beta_j}\left(\frac{Z_q^{1-q} - 1}{1 - q}\right). \qquad (6)$$

The reason to add the constant $[-1/(1 - q)]$ will become clear in what follows.

The value of the q entropy for a system in thermal equilibrium, from (1) and (4), is

$$S_q(U_q^j; \lambda) = -\sum_j \beta_j \frac{\partial}{\partial \beta_j}\left(\frac{Z_q^{1-q} - 1}{1 - q}\right) + \left(\frac{Z^{1-q} - 1}{1 - q}\right), \qquad (7)$$

where we have indicated the "natural" independent variables on which the q entropy depends. According to (6) and (7), S_q is the Legendre transform in the variables β_j of the state function

$$-\frac{Z^{1-q} - 1}{1 - q} \equiv \sum_k \beta_k F_q^k(\beta_j; \lambda). \qquad (8)$$

The functions F_q^k generalize the free energy of the system, and for the limit $q \to 1$ we recover $\sum_k \beta_k F_1^k = -\ln Z_1$. From (7) we immediately get

$$\beta_j = \frac{\partial S_q}{\partial U_q^j}. \qquad (9)$$

We notice from (4) that the invariance of ρ_s with respect to a homogeneous linear transformation that leads from the set U^j to a set $U^{j'}$ implies that β_j transforms contragradiently to U^j. We shall refer to the parameters β_j as the *generalized temperature* of the system. We shall apply the present formalism to discuss the cosmic blackbody radiation in a relativistic model of the Universe within the framework of the nonextensive thermostatistics.

Let us present first the extended forms of the first and second laws of thermodynamics. The generalized concept of performance of work on the system is related to the change in the q-expectation value of the observables U^j when the parameters λ change. We obtain in this way what is called the adiabatic change [2] of U_q^j,

$$\delta_{ad} U_q^j = \mathrm{tr}\left(\rho^q \frac{\partial U^j}{\partial \lambda}\right)\delta\lambda. \qquad (10)$$

The generalized "transfer of heat" is then introduced through the difference between the total change of the q-expectation value of U^j and (10),

$$\delta_Q U_q^j = \delta U_q^j - \delta_{ad} U_q^j. \qquad (11)$$

From (7), (6), and (4), we immediately find that

$$\delta S_q = \sum_j \beta_j \delta_Q U_q^j. \qquad (12)$$

Expressions (11) and (12) are then the generalized forms of the first and second laws of thermodynamics, for reversible processes, as obtained in the framework of a nonextensive thermostatistics and invariant under the symmetry group of ρ given by (4). We obtain a relativistic generalization in the particular case in which the symmetry group is the Lorentz group or a general coordinate transformation. All the quantities are to be considered at a particular spacetime point.

Let us return now to the cosmic blackbody radiation. We isolate, at a given spacetime point, a pencil of electromagnetic radiation (a light beam), characterized by a frequency and direction of propagation. All this information is contained in a null vector k $(k \cdot k = 0)$. In a local frame of reference we write $k = \{k^\mu\}$. The corresponding four-momentum operator is $p^\mu(\lambda) = \hbar k^\mu(\lambda) n$, where n is the number operator whose eigenvalues are nonnegative integers. The set of operators $U^j(\lambda)$ are given by $p^\mu(\lambda)$. The external parameters λ are the frequency and direction of propagation of the light beam. They may be identified with the wave vector $\vec{k}$. Let us now suppose that we are in a region of space that is filled with radiation. A straightforward generalization, for any value of q, of the results obtained by Hamity [3] will show that if the radiation is in thermal equilibrium we can choose the different world vectors β_μ, of the various pencils of radiation passing through the point, to have the same value. On the other hand, from the requirement of having n_q finite and positive definite for all values of $\vec{k}$, and considering the expression for the generalized Planck law [9], we have that β_μ must be timelike and pointing towards the future; i.e., there exists a (local) frame of reference in which β_μ will only have the time component different from zero: $\beta_\mu = (\beta, 0, 0, 0)$. The parameter β may be identified with $1/T$, where T is the absolute temperature assigned to the radiation. In order to compute n_q and S_q we need the generalized partition function $Z_q(\beta_\mu, \vec{k})$. The calculation of n_q in the limit $\beta(1 - q) \to 0$ and the comparison of the results with the Boltzmann-Gibbs statistics are given by Tsallis, Sa Barreto, and Loh [9] (see also Plastino, Plastino, and Vucetich [10]).

We wish to end our discussion with a brief comment on the thermal history of the cosmic blackbody radiation in a relativistic model of the Universe. We notice that the generalized partition function depends on β_μ and $\vec{k}$ through the quantity $\beta_\mu k^\mu$. Consider any homogeneous isotropic model of the Universe (a Robertson-Walker model) [12] and the isotropic observers with world velocity u, at "rest" with respect to the cosmic blackbody radiation. Then, in a frame of reference defined by u, we have

$$\beta^\mu = \frac{u^\mu}{T}. \qquad (13)$$

Since k is a null vector, we have

$$k \cdot u = -\frac{k \cdot \xi}{(\xi \cdot \xi)^{1/2}}, \quad (14)$$

where ξ is a Killing vector field which points in the direction of the propagation of k into the spacelike hypersurface of homogeneity [12]. The length of ξ varies from point to point in the spacetime and is proportional to the length scale factor of the Universe

$$\frac{(\xi \cdot \xi)_1^{1/2}}{(\xi \cdot \xi)_2^{1/2}} = \frac{a(\tau_1)}{a(\tau_2)}. \quad (15)$$

On the other hand, we have that $\xi \cdot k$ is constant along the geodesic with tangent vector *k;* also, since the free propagation of a light beam is a reversible process, the q-expectation value n_q is constant [it is straightforward to verify that (12) becomes in this case $\delta S_q = \hbar \delta n_q \sum_\mu \beta_\mu k^\mu$ [3]], and we have

$$(\beta \cdot k)_1 = (\beta \cdot k)_2 \Rightarrow \frac{T(\tau_1)}{T(\tau_2)} = \frac{a(\tau_2)}{a(\tau_1)}. \quad (16)$$

Thus, *the temperature of the cosmic blackbody radiation varies as the inverse of the scale factor of the Universe and is independent of q*, i.e., independent of the degree of nonextensitivity. The cooling in the course of the expansion is adiabatic. It is interesting to notice that although n_q is constant along a null ray the temperature vector β_μ changes from point to point. The use of the cosmic blackbody radiation to set observational bounds to q have been discussed by Plastino, Plastino, and Vucetich [10]. Although these bounds have been judiciously computed, we do not agree with the authors' conclusions that they found no violation of extensitivity in a *large scale* on the basis of these results. As our calculations show, the application of thermodynamics to the cosmic blackbody radiation is *strictly local*, and what we do in a large scale is just to compare intrinsic parameters, such as the temperature at different epochs. Of course, the blackbody spectrum depends on the parameter q, and it has been approximately calculated by Tsallis, Sa Barreto, and Loh [9]. We hope the present considerations enlighten the physical interpretation of experimental data such as those obtained with the COBE (Cosmic Background Explorer), and will stimulate further experiments and analysis.

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[1] D. van Dantzig, Physica (Utrecht) **6**, 673 (1939); Proc. Koninkl. Ned. Akad. Wetenschap. **42**, 601 (1939); **42**, 608 (1939).
[2] P. G. Bergmann, Phys. Rev. **84**, 1026 (1951).
[3] V. H. Hamity, Phys. Rev. **187**, 1745 (1969); see also Nature (London) **226**, 497 (1970).
[4] E. M. F. Curado and C. Tsallis, J. Phys. A **24**, L69 (1991); **24**, 3187(E) (1991); **25**, 1019(E) (1992).
[5] D. H. Zanette and P. A. Alemany, Phys. Rev. Lett. **75**, 366 (1995); C. Tsallis, S. V. F. Levy, A. M. C. Souza, and R. Maynard, Phys. Rev. Lett. **75**, 3589 (1995).
[6] G. Kaniadakis, A. Lavagno, and P. Quarati, Phys. Lett. B **369**, 308 (1996).
[7] B. M. Boghosian, Phys. Rev. E **53**, 4754 (1996).
[8] D. A. Stariolo and C. Tsallis, in *Optimization by Simulated Annealing: Recent Progress*, Ann. Rev. Comp. Phys., Vol. II, edited by D. Stauffer (World Scientific, Singapore, 1995), p. 343.
[9] C. Tsallis, F. C. Sa Barreto, and E. D. Loh, Phys. Rev. E **52**, 1447 (1995).
[10] A. R. Plastino, A. Plastino, and H. Vucetich, Phys. Lett. A **207**, 42 (1995).
[11] C. Tsallis, J. Stat. Phys. **52**, 479 (1988).
[12] R. M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984), Chap. 5.