

Random Matrix Model of QCD at Finite Density and the Nature of the Quenched Limit

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We use a random matrix model to study chiral symmetry breaking in QCD at finite chemical potential μ . We solve the model and compute the eigenvalue density of the Dirac matrix on a complex plane. A naive “replica trick” fails for $\mu \neq 0$; we find that quenched QCD is not a simple $n \rightarrow 0$ limit of QCD with n quarks. It is the limit of a theory with $2n$ quarks: n quarks with original action and n quarks with conjugate action. The results agree with earlier studies of lattice QCD at $\mu \neq 0$ and provide a simple analytical explanation of a long-standing puzzle. [S0031-9007(96)00467-X]

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The spontaneous breaking of chiral symmetry is one of the most important dynamical properties of QCD which shapes the hadronic spectrum. A great deal of understanding of this nonperturbative phenomenon at zero and finite temperature has been achieved by various methods [1]. In particular, we expect that the chiral symmetry is restored above a certain critical temperature. The study of this new chirally symmetric phase of hot QCD is one of the primary objectives of heavy ion colliders. In contrast, the behavior of QCD at large baryon density (conditions which can arise in the heavy ion colliders or in neutron stars) is not well understood. The main puzzle has for a long time been a contradiction between a straightforward physical expectation and numerical results from quenched lattice QCD [2,3]. Simulations with dynamical quarks, on the other hand, are very inefficient at finite μ ; the fermion determinant is complex.

The puzzle concerns the dependence of the order parameter (the chiral condensate $\langle \bar{\psi}\psi \rangle$) on the baryon chemical potential. A nonanalytical change in the value of $\langle \bar{\psi}\psi \rangle$ should occur when $\mu > \mu_c \approx m_B/3$, where m_B is the mass of the lightest baryon. At this point the production of baryons becomes energetically favorable. For smaller μ the value of $\langle \bar{\psi}\psi \rangle$ is nonzero. In contrast, lattice simulations of quenched QCD indicate that $\mu_c = 0$ (at zero bare quark mass); i.e., the chiral condensate vanishes if $\mu \neq 0$ [2,3]. A number of possible explanations have been suggested [4]. However, the answer to this puzzle remains unclear.

This work was motivated by a desire to shed some light on this question using the random matrix approach which received considerable interest recently [5–12]. It is based on the idea that, for the purpose of studying chiral symmetry breaking, fluctuations of the Dirac operator in the background of the gauge fields can be approximated by purely random fluctuations of its matrix elements in a suitable basis. For example, in the instanton liquid model this basis can be formed from the Dirac zero modes for individual (anti)instantons, which due to overlaps form a band of small eigenvalues responsible for the chiral symmetry breaking [13]. A similar ran-

dom matrix approach is fruitful in the studies of spectra of systems with a high level of disorder, such as spectra of heavy nuclei [14]. Introduction of chemical potential into such a model of chiral symmetry breaking is straightforward. The resulting Dirac matrix (times i) is *non-Hermitian*. Thus, the eigenvalues lie in the complex plane rather than on a line. Such random matrix models have not received much attention previously, and this study is a step in an unexplored direction.

In this Letter we show how to solve such a model in the thermodynamic limit and discuss the implications.

In order to study chiral symmetry breaking, we shall calculate the resolvent of the Dirac operator D :

$$G = \langle \text{tr}(z - D)^{-1} \rangle, \quad (1)$$

as a function of the bare quark mass z which we take to be a complex variable $z = x + iy$. The average is over fluctuations of the random matrix elements of D . It should be obvious that G is the same as $\langle \bar{\psi}\psi \rangle$. The resolvent can be expressed through the average eigenvalue density ρ :

$$G(x, y) = \int dx' dy' \rho(x', y') \frac{1}{z - z'}. \quad (2)$$

A vector $\vec{G} = (\text{Re}G, -\text{Im}G)$ is the electric field created by the charge distribution ρ . This makes the inversion of (2) obvious:

$$\rho = \frac{1}{2\pi} \vec{\nabla} \vec{G} = \frac{1}{\pi} \frac{\partial}{\partial z^*} G, \quad (3)$$

where $\partial/\partial z^* \equiv (\partial/\partial x + i\partial/\partial y)/2$.

Analytical properties of G are very closely related to the chiral symmetry breaking. From (3) we see that ρ vanishes if the function G is holomorphic. A discontinuity of G along a cut going through $z = 0$ is the signature of the spontaneous chiral symmetry breaking: $\langle \bar{\psi}\psi \rangle(+0) \neq \langle \bar{\psi}\psi \rangle(-0)$. This observation together with (3) leads to the Banks-Casher relation [15]: $\langle \bar{\psi}\psi \rangle = \pi\rho(0)$, where $\rho(0)$ is the density per *length* on the cut at

$z = 0$. However, (3) is more general and can be applied to a case when the nonanalyticity is not in the form of a cut, but occupies a two-dimensional patch, which is the case in our model.

The matrix D has the form

$$D = \begin{pmatrix} 0 & iX \\ iX^\dagger & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad (4)$$

where we added the chemical potential term $\mu\gamma_0$ to the Dirac matrix [5]. The $N \times N$ matrix elements of X are independently distributed complex Gaussian random variables: $P(X) = \text{const} \times \exp\{-N \text{Tr}XX^\dagger\}$. The unit of mass in the model is set by $n_4/\langle\bar{\psi}\psi\rangle_0 \sim 200$ MeV, where $n_4 \sim 1 \text{ fm}^{-4}$ is the number of small eigenvalues in a unit volume (instanton density [16]) and $\langle\bar{\psi}\psi\rangle_0 \sim (200 \text{ MeV})^3$ is the chiral condensate at $T = 0$, $\mu = 0$.

In order to find the resolvent (1), we introduce n quark fields (replicas) and calculate

$$V_n = -\frac{1}{n} \ln\langle \det^n(z - D) \rangle. \quad (5)$$

This quantity continued to $n \rightarrow 0$ (quenched limit) becomes

$$V = -\langle \ln \det(z - D) \rangle, \quad (6)$$

from which we find G :

$$G = -\frac{\partial}{\partial z} V. \quad (7)$$

The trace in (1) is normalized as $\text{tr}1 = \text{Tr}1/(2N)$. Following the electrostatic analogy of the previous section, one can view $\text{Re}V$ as the scalar potential for \vec{G} .

Using Hubbard-Stratonovitch transformation we obtain

$$\exp\{-nV_n\} = \int \mathcal{D}a \det^N \begin{pmatrix} z + a & \mu \\ \mu & z + a^\dagger \end{pmatrix} \times \exp\{-N \text{Tr}aa^\dagger\}, \quad (8)$$

where a is an auxiliary complex $n \times n$ matrix field. For large N the calculation of the integral amounts to finding its saddle point. If we assume that the replica symmetry is not broken (i.e., a is proportional to a unit matrix), we arrive at the saddle point equation:

$$(z + a) = a[(z + a)^2 - \mu^2], \quad (9)$$

The complex value of a in (9) is the analytical continuation of the real part of the diagonal matrix elements of a in (8). The imaginary part is zero in the saddle point. The solution of this cubic equation is straightforward. It is the same as in a similar model [10] with $\omega \rightarrow i\mu$. Finally, it is easy to find using (7) and (8) that $G = a$ where a is the saddle point given by (9).

The V_n does not depend on n , and the limit $n \rightarrow 0$ seems obvious. However, below we shall compare this expectation to numerical data and see that the limit $n \rightarrow$

0 is in fact very different. Now let us summarize the properties of this model for $n > 0$.

We see that $G(z)$ is a holomorphic function. It has three Riemann sheets, and we select the one where $G \rightarrow 1/z$ for $z \rightarrow \infty$, which follows from $\int dx dy \rho = 1$ and the Gauss theorem. The only singularities on the physical sheet are the pole at $z = \infty$ and 2 (for $\mu^2 \leq 1/8$) or 4 (for $\mu^2 > 1/8$) branch points connected by cuts. The branch points are where two of the three solutions of (9) coincide. The trajectory of a cut is determined by a condition that a is the deepest minimum (out of 3) of $\text{Re}\{a^2 - \ln[(z + a)^2 - \mu^2]\}$.

At $\mu = 0$ the cut along the imaginary axis connects two singularities at $z = \pm 2i$. For nonzero μ the singularities start moving toward each other along the imaginary axis. At $\mu^2 = 1/8$ each of the branch points bifurcates in two pieces which move off the y axis into the complex plane. The cut goes through the origin (along the y axis) until $\mu^2 = 0.278, \dots$. At this point it splits into two cuts connecting complex conjugate points. This means that $\mu_c^2 = 0.278, \dots$ in such a model.

For $n = 0$ one can easily determine the density of eigenvalues numerically by calculating the eigenvalues of the random matrix D and plotting them on a complex plane. The density of points on such a scatter plot is proportional to ρ . The results for different values of μ^2 are shown in Fig. 1. They contradict naive expectations from the previous section. At $\mu = 0$ all eigenvalues are distributed between points $z = \pm 2i$ on the y axis. However, already at very small nonzero $\mu^2 \ll 1/8$, the eigenvalue density is nonzero in a "blob" of finite width in the x direction which grows with μ . The same behavior is seen in quenched lattice QCD [2] and gives rise to the paradox described earlier: there is no discontinuity in the value of $\langle\bar{\psi}\psi\rangle$ at any $\mu > 0$. The matrix model has an advantage: it is amenable to exact treatment which clarifies the nature of the problem.

The failure of the naive replica approach can be understood if we look at the expression (5): it does not contain z^* . On the other hand, Eq. (3) tells us that $\rho \neq 0$ if G depends on z^* , i.e., if it is not holomorphic. In fact, the correct replica trick for a non-Hermitian matrix should start from the quantity

$$V_{n,n} = -\frac{1}{n} \ln\langle \det^n(z - D)(z^* - D^\dagger) \rangle, \quad (10)$$

which is now real due to the introduction of the quarks with conjugate Dirac matrix. Naively, in the limit $n \rightarrow 0$ the conjugate quarks decouple but, as we shall see, this is not always the case. In mathematics an analogous construction is called a V-transform [17] and allows one to study spectra of non-Hermitian matrices. In the present context this formal construction has a clear and simple physical meaning.

We can calculate (10) using the same method as for (5). Now, however, we have to introduce four auxiliary

complex $n \times n$ fields, and we arrive at

$$\exp(-nV_{n,n}) = \int \mathcal{D}a \mathcal{D}b \mathcal{D}c \mathcal{D}d \times \det^N \begin{pmatrix} z + a & \mu & 0 & id \\ \mu & z + a^\dagger & ic & 0 \\ 0 & id^\dagger & z^* + b^\dagger & \mu \\ ic^\dagger & 0 & \mu & z^* + b \end{pmatrix} \times \exp\{-N(|a|^2 + |b|^2 + |c|^2 + |d|^2)\}. \quad (11)$$

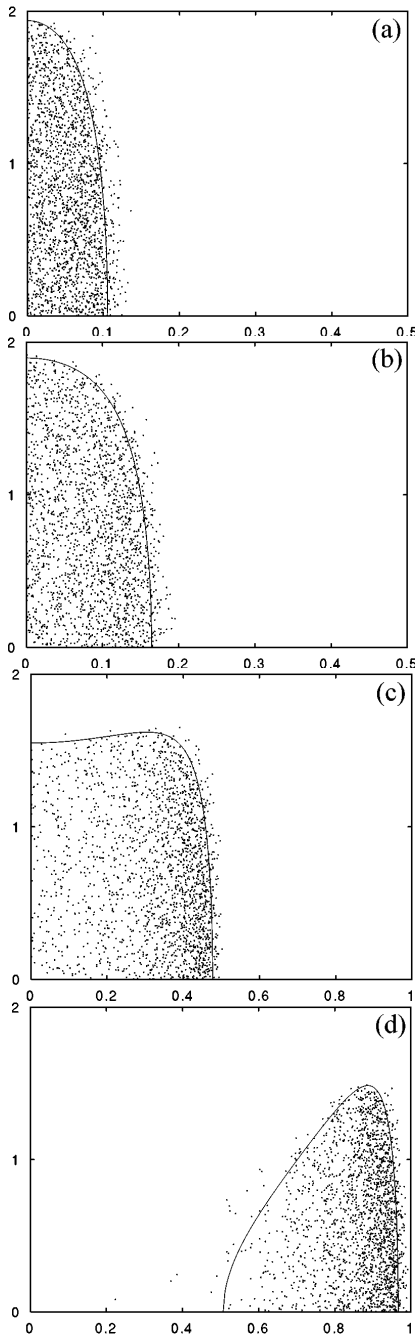


FIG. 1. Scatter plots on a complex plane (x,y) of the eigenvalues of an ensemble of 20 random 100×100 matrices D at four values of μ^2 : 0.06 (a), 0.10 (b), 0.40 (c), and 1.20 (d). The solid curves follow Eq. (12).

The set of solutions of the saddle point equation is richer in this case. There is a solution with $c = d = 0$. In this case the conjugate quarks do decouple and we obtain the same holomorphic function G as before. However, there is another solution in which the condensates c and d are not zero. Then the function G is not holomorphic and therefore $\rho \neq 0$. This saddle point dominates the integral at small z for $0 < \mu < 1$.

The condensates c and d are bilinears of the type $\langle \bar{\psi} \chi \rangle$, mixing original ψ and conjugate χ quarks. These condensates do not break the original chiral symmetry but a spurious (replica type) symmetry involving both original and conjugate quarks. Similar condensates carrying baryon number were discussed in the SU(2) model of QCD with quarks [18]. In the quenched theory, as in [18], the original chiral symmetry is always restored at $\mu > 0$. The spurious symmetry is spontaneously broken for $\mu < 1$ and is restored for $\mu > 1$.

The boundary of the $\rho \neq 0$ region is given by

$$y^2 = (\mu^2 - x^2)^{-2} [4\mu^4(1 - \mu^2) - (1 + 4\mu^2 - 8\mu^4)x^2 - 4\mu^2x^4]. \quad (12)$$

It is plotted in Fig. 1 for comparison with numerical data. The baryonic condensates c and d inside of the “blob” are given by

$$|c|^2 = |d|^2 = \frac{\mu^2}{\mu^2 - x^2} - \mu^2 - \frac{x^2}{4(\mu^2 - x^2)^2} - \frac{y^2}{4}. \quad (13)$$

On the boundary (12) they vanish and the two solutions (holomorphic and nonholomorphic) match. In the outer region, $c = d = 0$ and $G = a$ is the solution of the cubic equation (9). Inside of the “blob” the resolvent is given by

$$G = a = \frac{1}{2} \frac{x}{\mu^2 - x^2} - x - \frac{iy}{2}, \quad (14)$$

and the density of the eigenvalues (3) is

$$\rho = \frac{1}{4\pi} \left(\frac{x^2 + \mu^2}{(\mu^2 - x^2)^2} - 1 \right). \quad (15)$$

To appreciate the nontriviality of this result, one should notice that expression (1) which defines the resolvent appears to depend only on z . The limit $n \rightarrow 0$ must be taken with great care, as is well known in the replica approach [19].

In conclusion, the fermion determinant in QCD is complex at nonzero chemical potential. Lattice simulations of such a theory are extremely inefficient. Therefore, all reliable data from lattice QCD so far have been obtained for a quenched theory. We learn from the random matrix model that the quenched theory at finite μ behaves qualitatively different from the QCD with dynamical quarks. Rather, the quenched approximation describes a theory where each of the quarks has a conjugate partner, so that

the fermion determinant is non-negative. We see that for such a theory the result $\mu_c = 0$ is natural. Similar arguments have been given by several authors in different settings and using less realistic models [4]. Here it can be demonstrated in a very clean and explicit way.

Simulations with dynamical quarks at strong coupling are possible in SU(2) and SU(4) QCD [18] and also agree with our results. The μ_c is finite in the SU(4) theory. On the other hand, in the SU(2) theory, where the quarks are self-conjugate, $\mu_c = 0$ due to the baryonic condensates.

The matrix model describes many features of the chiral symmetry breaking in QCD very well [5–12]. One of the apparent limitations, however, is that it is static—there are no kinetic terms and we cannot study the spectrum of masses. In the quenched QCD μ_c appears to coincide with half of the mass of the so-called baryonic pion [4]—a bound state of a quark and a conjugate antiquark. It is degenerate with the π meson, but carries a nonzero baryon number. From the exact solution (12) we find $\mu_c \approx \sqrt{m/2}$, for small quark mass $m \ll 1$. If we had f_π in our model we could relate μ_c to the mass of the pion. The model also does not account for the confinement of quarks. It remains to be seen if the confinement plays a role in the case under consideration.

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