Phase Structure of Systems with Multiplicative Noise

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The phase diagrams and transitions of nonequilibrium systems with multiplicative noise are studied theoretically. We show the existence of both strong- and weak-coupling critical behavior, of two distinct active phases, and of a nonzero range of parameter values over which the susceptibility is infinite in any dimension. A scaling theory of the strong-coupling transition is constructed. [S0031-9007(96)00273-6]

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Though they have been argued [1] to describe a diverse and important set of physical systems out of equilibrium, among them autocatalytic chemical processes and several different problems in quantum optics, dissipative partial differential equations (PDE's) with multiplicative noise remain poorly understood. Such equations typically admit two types of phases: trivial or "absorbing" phases in which the dynamical variable(s) or order parameter(s) vanish identically at all points in space, and remain zero in perpetuity; and nontrivial or "active" phases in which these variables have nonzero expectation values. To date, much of the theoretical effort [1,2] on systems of this type has been devoted to calculating the critical value of the control parameter at which the continuous transition between these two phases occurs [3].

In this paper we present a somewhat broader investigation of the multiplicative-noise problem, along the lines of existing analyses of critical phenomena in the directed percolation problem [4], and in the many physically relevant stochastic PDE's with additive noise. In particular, we analyze the phase structure and critical properties of multiplicative-noise problems with different symmetries. We show that above an upper critical dimension, $d_c = 2$, the transition can belong to one of two distinct universality classes described by different fixed points: a weak coupling, mean-field fixed point, accessible for noise strength D less than a (nonuniversal) critical value D_c; and a strong coupling fixed point with nontrivial exponents, accessible for $D > D_c$. The separatrix at $D = D_c$ is described by a third, unstable, nontrivial fixed point. For $d \leq 2$, only the strong coupling critical behavior occurs. We analyze the mean-field transition, compute the critical exponents on the separatrix in an expansion in $\epsilon \equiv d - 2$, and formulate a scaling description of the strong coupling transition, which we check against previous exact results for $d = 0$ (the single-variable problem). We also demonstrate the occurrence of a rather striking phenomenon: For discrete symmetry and any d, there is a region of the phase diagram in which the response of the system to an infinitesimal uniform field (i.e., the uniform susceptibility), is everywhere infinite. For systems with N-component order parameters and $O(N)$ symmetry with $N \geq 2$, we argue for the existence of two distinct active phases for $d > 2$, one which breaks the $O(N)$ symmetry and one which does not. We illustrate this with an exact calculation for $N = \infty$.

The models we study are defined by the equation

$$\partial_t n_\alpha(\vec{x}, t) = \mu \nabla^2 n_\alpha - r n_\alpha - \frac{u|\vec{n}|^2 n_\alpha^\rho}{N} + |\vec{n}|\eta_\alpha . \quad (1)$$

Here $\vec{n} \equiv \{n_1, \ldots, n_N\}$ is an N-component, real vector field, r and $u(> 0)$ are real parameters, and $\vec{\eta}$ is a Gaussian noise vector with correlations $\langle \eta_\alpha(\vec{x}, t) \times \eta_\beta(\vec{x}', t')\rangle = D\delta_{\alpha\beta}\delta(\vec{x} - \vec{x}')\delta(t - t')/N$ and noise strength D. We deal primarily with the cubic nonlinearity $\rho = 1$, though for the single-component theory, $N = 1$, we also discuss the quadratic nonlinearity, $\rho = 0$, relevant for, e.g., chemical reactions [1]. We work in the Ito [5] representation.

We first discuss mean-field theory [2], implemented by dropping the noise term in (1), and ignoring spatial variations in $\vec{n}(\vec{x}, t)$, which is replaced by $\vec{n}(t)$. For $N = 1$, Eq. (1) then has two steady-state solutions, representing the absorbing and active phases: $n(t) = 0$, stable for $r > 0$, and $n = (-r/u)^{1/(1+\rho)}$, stable for $r < 0$. For the continuous symmetry case $N \geq 2$ [where only the cubic nonlinearity, $\rho = 1$, preserves the $O(N)$ symmetry and so need be considered] a result identical to that for $N = 1$, $\rho = 1$ holds, except that in the active phase the orientation of $\vec{n}$ is unspecified, the magnitude being given by $\vec{n}^2 = -r/u$. In both cases the transition between the phases occurs at $r = r_c = 0$, and the critical exponent β characterizing the vanishing of n as $r \to r_c$ takes the value $\beta = 1/(1 + \rho)$, independent of dimension, d. The correlation length and dynamical exponents assume their usual mean-field values, $\nu = 1/2$ and $z = 2$, respectively. (Alternatively, the mean-field results can be obtained [2] from a model with infinite-range interactions, or, equivalently, $d = \infty$.)

It is easy to determine the critical dimension d_c above which this mean-field result is valid, by considering the absorbing phase where $\langle \vec{n}(\vec{x}, t)\rangle = 0$. In this phase, one can calculate certain correlation and response functions exactly, by summing perturbation expansions to all orders [6]. We first specialize to $N = 1$, where the exact renormalized noise, D_R, and coupling constant, u_R, are given by $D_R = DA$ and $u_R = uA$, with $A = 1/[1 - DI_d(r)]$, and $I_d(r) \equiv \int [d^d k/(2\pi)^d][1/2(\mu k^2 + r)]$. Note that $I_d(r)$ is finite for all $r \geq 0$, provided $d > 2$ [7]. Hence for small enough D the denominator of A remains bounded below by a positive number, even when r decreases to zero. Thus D_R and u_R remain finite and nonzero throughout the absorbing phase, right down to the critical point, which continues to occur precisely at $r = 0$, provided D is small enough and $d > 2$. In this regime, all other quantities of interest likewise experience no singular renormalization near $r = 0$, so the mean-field critical point at $r = 0$ and mean-field exponents remain exact. [In fact, some quantities, such as the response function $g(\vec{k}, \omega) = (-i\omega + \mu k^2 + r)^{-1}$, experience no diagrammatic renormalization in the absorbing phase [6].]

For $d \leq 2$, by contrast, $I_d(r)$ diverges at $r = 0$, so D_R and u_R both blow up for some *positive* value of r, even for infinitesimal D. This invalidates mean-field theory, and suggests that the critical value r_c is shifted away from zero, and that the mean-field critical exponents can no longer be trusted. The same is true for $d > 2$, provided D exceeds the special value D_c for which D_R first diverges at $r = 0$. These results strongly suggest that $d_c = 2$, but that even for $d > d_c$ there may be nontrivial critical behavior at large D [8].

This expectation is confirmed by straightforward renormalization-group (RG) analysis [9] of models (1), in the absorbing phase and at the critical point. Imagine rescaling space, time, and the field according to $\vec{x} = b\vec{x}'$, $t = b^z t'$, and $\vec{n}(\vec{x}, t) = b^\zeta \vec{n}'(\vec{x}', t')$, where z and ζ are as yet undetermined exponents. Writing $b = e^l$, using standard methods and the results quoted above, one readily derives the following recursion relations:

$$\begin{aligned} d\mu/dl &= (z - 2)\mu\,, \\ dr/dl &= 2r\,, \\ du/dl &= u[(1 + \rho)\zeta + z + (1 + 2\rho)A_d D] \\ &\quad + O(\rho u^2), \\ dD/dl &= D(z - 2 - \epsilon + A_d D). \end{aligned} \tag{2}$$

Here $\epsilon = d - 2$, and $A_d = 1/4\pi + O(\epsilon)$ is a positive, d-dependent constant. Owing to the aforementioned absence of any diagrammatic renormalization of the response function in the absorbing phase, the μ and r equations are exact. The u and D equations contain no further diagrammatic corrections in the case $\rho = 0$. For $\rho = 1$, there are, as indicated in (2), corrections to the u equation of $O(u^2)$.

Finite fixed points of these recursion relations can only be reached by choosing $z = 2$ and tuning the initial value of r, to zero. Choosing $\zeta = -[z + (1 + 2\rho)A_d D]/(1 + \rho)$ maintains the u equation at a fixed point with $u^* = u(l = 0)$. Then one need only look for stable fixed points of the D equation.

For $d > 2$ ($\epsilon > 0$), this equation has a stable "weak-coupling" fixed point at $D^* = 0$, which supplements the other fixed point values: $r^* = 0$ and $u^* = u(l = 0)$. This fixed point gives rise to mean-field exponents (e.g., $z = 2$ and $\nu = 1/2$ follow immediately from the nonrenormalization of the response function), and is reached for initial values of D less than $D_c = \epsilon/A_d$.

For $D > D_c$, however, the recursion relation for D runs off to $D = \infty$, rendering the critical behavior incalculable by perturbative techniques. Presumably the transition is controlled by a nonperturbative "strong-coupling" fixed point in this regime. For $d \leq 2$, however, the weak-coupling fixed point with $D^* = 0$ is unstable for *any* positive D, and strong-coupling behavior always obtains.

While the strong-coupling transition cannot yet be described in full detail [10], certain aspects of it have been elucidated in previous work. In particular, Becker and Kramer [2] have derived results for the amount, r_c, by which the critical value of r is shifted away from $r = 0$ at this transition. They find that $r_c < 0$ (in the Ito representation) for $d = 1$ and $d = 2$. There is also a rather complete solution [1,3] of the single-variable ($d = 0$) problem for $N = 1$, where r_c is also shown to be negative. These results allow one to make the striking prediction that for $N = 1$ the susceptibility of model (1) at zero frequency ($\omega = 0$) and wave vector ($\vec{k} = 0$) diverges over some nonzero range of values of r. To understand this, first recall that, as discussed above, the response function $g(\vec{k}, t)$ in the absorbing phase does not undergo any diagrammatic correction due to the nonlinearity, and so is given by the linear result $g(\vec{k}, t) = \theta(t)e^{-(\mu k^2 + r)t}$. It follows at once that the susceptibility at $\vec{k} = \omega = 0$, defined as $\chi = \int_0^\infty dt\, g(\vec{k} = \vec{0}, t)$, is infinite for all negative values of r that lie in the absorbing phase. The previous results that $r_c < 0$ for $d \leq 2$ (which we extend to $d > 2$ below), therefore imply the divergence of χ in the entire range $0 \geq r \geq r_c$. For $d = 0$ we demonstrate explicitly below that this range is controlled by a fixed line with a continuously varying exponent. Note that the critical exponents z and ν associated with the response function take their mean-field values, 2 and 1/2, respectively, at the point $r = 0$ where the susceptibility diverges. This is consistent with the recursion relations for r and μ in (2). Keep in mind, however, that the strong-coupling transition into the active phase occurs at $r(l = 0) = r_c < 0$, and so is represented by an inaccessible fixed point of the r recursion relation with $r^* = -\infty$.

Thus for $N = 1$ we are led to the schematic phase diagrams shown in Fig. 1. In constructing the diagram for $d > 2$, we have generalized the calculation of Ref. [2(b)] of r_c to dimensions larger than 2. That calculation is based on mapping the computation of r_c onto the quantum mechanical problem of finding the lowest bound state energy of a potential given by the spatial correlation function of the noise. The analysis can easily be extended to dimensions $d > 2$, where it is well known that the depth of a potential well has to exceed a d-dependent critical value, D_c, in order for there to be a bound state. The existence of a bound state for $D > D_c$ implies that the critical value of r is shifted to $r_c < 0$, corresponding to the strong-coupling fixed point. When $D < D_c$, there is no bound state, so $r_c = 0$, corresponding to the weak-coupling fixed point. These results, though obtained in a different way, are fully consistent with our earlier RG analysis, and are summarized in Fig. 1(b). The critical exponents of the multicritical point P in Fig. 1(b) are readily found to be $z = 2$, $\nu = 1/2$, and $\beta = [2 + (1 + 2\rho)\epsilon]/2(1 + \rho) + O(\rho\epsilon^2)$.

We now turn to the phase diagrams of the $O(N)$ symmetric models with $N \geq 2$ and $\rho = 1$. The continuous symmetry of these models allows for two distinct active phases: one which preserves this symmetry [11] and one which breaks it. The first of these has $\vec{M} \equiv \langle \vec{n} \rangle = 0$ and $Q \equiv \langle q \rangle \equiv \langle (\vec{n})^2/N \rangle \neq 0$, while the second has both $\vec{M} \neq 0$ and $Q \neq 0$. Since for $d \leq 2$ it is extremely difficult to break a continuous symmetry in a noisy system [12], we anticipate that the only active state is the symmetric one in this case. For $d > 2$, both active phases can occur in the phase diagram. The absorbing phase always occurs for sufficiently large r.

These features can be studied explicitly in the exactly solvable limit of $N = \infty$, where all diagrams contributing to the perturbation expansions for quantities of interest can be easily summed, with the following results.

For $d \leq 2$ there is, as anticipated, no phase that breaks the $O(N)$ symmetry, i.e., for which $\vec{M} \neq 0$. There is, however, a symmetric active phase with $Q > 0$. The value of Q is determined by $DI_d(r + uQ) = 1$ (where I_d was defined earlier). The phase boundary $r_c(D)$ between the absorbing and active phases is thus determined by $DI_d(r_c) = 1$, so that $r_c(D) \sim D^{-2/\epsilon}$ as $D \to 0$. Note that r_c is *positive* here. Hence the susceptibility at $\vec{k} = \omega = 0$ remains finite at the transition, reflecting the fact that $\langle \vec{n} \rangle = 0$ in the active phase. Critical exponents are, of course, also readily calculated. For example, the order parameter Q vanishes like $(r_c - r)^\beta$ with $\beta = 1$.

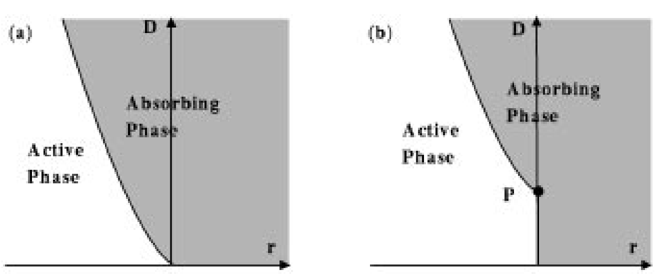

FIG. 1. Schematic phase diagram for model (1) with $N = 1$ for (a) $d \leq 2$, (b) $d > 2$. Weak- (strong-) coupling transition occurs below (above) the multicritical point P in (b); transition is always strong coupling in (a). Susceptibility diverges in the absorbing phases when $r \leq 0$, and possibly also in portions of the active phases.

For $d > 2$ the phase diagram is somewhat more complicated, containing a broken-symmetric phase for small D, a symmetric active phase for larger D [see Fig. 2(b)], and a multicritical point where these active phases and the absorbing phase meet. Again, all phase boundaries and critical exponents can be exactly computed. For example, the transition from the absorbing phase to the broken-symmetry active phase is mean-field-like, and occurs at the unshifted critical value $r_c = 0$. The exponent governing the decay of Q at the symmetric-to-absorbing transition continues to assume the value $\beta = 1$, the phase boundary being determined by $DI_d(r_c) = 1$, which has a solution only for $D > D_c$, with $D_c = 1/I_d(0)$. The second-order phase boundary between the two active phases occurs at $D = D_c$ for all $r < 0$. The multicritical point S occurs at $r = 0$, $D = D_c$ [Fig. 2(b)].

While many details of these phase diagrams are doubtless special to $N = \infty$, certain qualitative features should continue to apply to physically relevant values of N such as $N = 2$. The existence of both types of active phases for $d > 2$, but only the symmetric active phase for $d \leq 2$, is, e.g., a general feature. For $d > 2$, the broken-symmetric active phase will continue to occur for small D, where the exponents for the transition to the absorbing phase continue to assume mean-field values, and $r_c = 0$. The exponents for the absorbing-to-symmetric active phase transition are presumably controlled by a strong-coupling fixed point, and so are difficult to calculate. The transition between the two active phases belongs in the universality class of the equilibrium $O(N)$ model.

We turn next to a scaling characterization of the strong-coupling transition for $N = 1$. This is very similar to standard scaling theories of equilibrium critical phenomena, but since one of the phases is absorbing, an additional independent exponent is required for a complete description of the phase transition. To understand this, consider the RG analysis of the steady-state two-point connected correlation function $C(\vec{x}, t) = \langle n(\vec{x}, t) n(\vec{0}, 0) \rangle_c$. The rescalings of $n, \vec{x}$, and t given above yield the familiar RG equation [9] $C(\vec{x}, t, \delta r) = b^{2\zeta} C(\vec{x}/b, t/b^z, \delta r b^{1/\nu})$, where $\delta r = r_c - r$. First consider the equal-time case, $t = 0$. Here we obtain $C(\vec{x}) \sim \delta r^{-2\zeta\nu} c(x/\xi)$, where $c(y)$ is a scaling function, and the

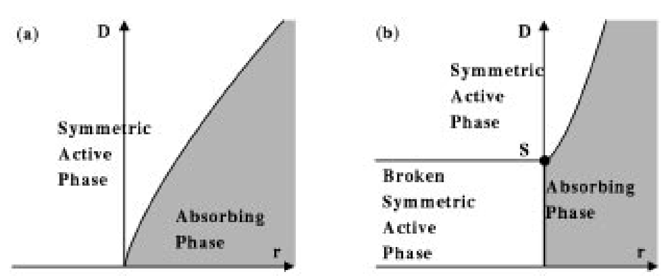

FIG. 2. Schematic phase diagram for model (1) with $N = \infty$ for (a) $d \leq 2$, (b) $d > 2$. Symmetric (broken-symmetric) to absorbing transition is strong (weak) coupling in (b); transition is always strong coupling in (a).

correlation length ξ diverges like $\xi \sim \delta r^{-\nu}$. In typical critical phenomena, one can go to the critical point, $\delta r = 0$, assume that $C(\vec{x})$ approaches a nonzero value in this limit, and conclude that $C(\vec{x}) \sim x^{2\zeta}$. Here, however, because C is identically zero in the absorbing phase, it vanishes as $\delta r \to 0$ from the active phase. Hence in the limit $\delta r \to 0$, $C(\vec{x})$ takes the form $\delta r^{\Delta} x^{-(d-2+\eta)}$, where Δ is a new, apparently independent, critical exponent, and η is the standard correlation function exponent. Consistency between this result and the previous scaling expression requires $c(y) \sim y^{-(d-2+\eta)}$ for $y \ll 1$, i.e., $\Delta = -\nu(2\zeta + d - 2 + \eta)$. This scaling law can be written in terms of the order parameter exponent β, shown above to satisfy $\beta = -\nu\zeta$, which yields $\beta = [\Delta + \nu(d - 2 + \eta)]/2$. This is a generalization of $\beta = \nu(d - 2 + \eta)/2$, which holds for ordinary equilibrium critical phenomena, where $\Delta = 0$. In the directed percolation problem [4], where the transition is also into an absorbing phase, a simple graphical argument (which fails for the current models), shows that $\Delta = \beta$, whereupon $\beta = \nu(d - 2 + \eta)$ [13].

Similar considerations allow one to easily generalize other scaling laws to account for the new exponent Δ. For example, the autocorrelation function $C(\vec{x} = 0, t)$ is readily shown to decay like $\delta r^{\Delta} t^{-(d-2+\eta)/z}$ as $\delta r \to 0$. The exponent γ governing the critical singularity of the equal-time correlation function (i.e., the static structure factor), at $\vec{k} = 0$ is given by $\gamma = \nu(2 - \eta) - \Delta$.

Finally, we verify some of the general results derived above in the solvable case of $d = 0$, $N = 1$. Let us first check the predicted divergence of the uniform susceptibility over the range $0 < r < r_c$, by solving the $d = 0$ problem in the presence of a uniform field, h, where the equation takes the form $dn/dt = -rn - un^{2+\rho} + h + n\eta(t)$. As in the case $h = 0$, the Fokker-Planck equation [5] for the steady-state probability distribution function $P(n)$ can be solved explicitly, with the (Ito representation) result $P(n) = \frac{1}{Z}\int_0^\infty dn\, n^{-2r/D-2} e^{-2h/Dn} e^{-2un^{\rho+1}/D(\rho+1)}$; here Z is a constant chosen to normalize $\int_0^\infty dn\, P(n)$ to unity. Given this expression, one readily calculates $\langle n \rangle$ as a function of h, thereby deriving a formula for the uniform susceptibility $\chi(h) \equiv \partial\langle n \rangle/\partial h$. In the limit of small h one finds that $\chi(h)$ approaches a finite limit as $h \to 0$, provided $|2r/D + 1| > 1$. In the band of r values defined by $|2r/D + 1| < 1$, however, $\chi(h = 0)$ is infinite, diverging like $h^{|2r/D+1|-1}$ as $h \to 0$. Since the transition from the absorbing to the active phase occurs [1,3] at $r_c = -D/2$, this is consistent with our general argument that $\chi(h = 0)$ is infinite throughout the range $0 < r < r_c$ whenever $r_c < 0$. The exact solution shows that for $d = 0$ the divergence actually extends into the active phase as well. The explicit dependence on r of the exponent governing this divergence as $h \to 0$ implies that the interval $|2r/D + 1| < 1$ is controlled by a fixed line of the RG, with a continuously varying exponent. It is easily shown that $\chi(h)$ diverges logarithmically at the ends of this interval, and has logarithmic corrections at the critical point $r = r_c$.

Next we use the $d = 0$ results to check the scaling theory developed above. In Ref. [3] one finds that the autocorrelation function $C(t)$ decays like $\delta r t^{-1/2}$ for $t \ll \tau$, where the characteristic time τ diverges like δr^{-2} for small δr. The order parameter decays like δr. According to our earlier definitions, these results imply exponent values $\Delta = 1$, $\beta = 1$, $\nu z = 2$, and $(d - 2 + \eta)/z = 1/2$, respectively. It is a trivial matter to verify that these values satisfy our proposed scaling relation $2\beta = \Delta + \nu(d - 2 + \eta)$.

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