Strange Nonattracting Chaotic Sets, Crises, and Fluctuating Lyapunov Exponents

Silvina Ponce Dawson*

Departamento de Física and Instituto de Astronomía y Física del Espacio, Facultad de Ciencias Exactas y Naturales, UBA, Ciudad Universitaria, Pabellón I, 1428 Buenos Aires, Argentina

(Received 26 April 1995; revised manuscript received 1 November 1995)

Chaotic attractors containing periodic orbits with different numbers of unstable directions display fluctuating Lyapunov exponents. We show that the existence of certain nonattracting chaotic sets inside the attractor guarantees the occurrence of this behavior in a persistent manner. These nonattracting sets can be brought inside the attractor via a new type of crisis and may be created, as a parameter is varied, via a sequence of bifurcations out of unstable periodic orbits. [S0031-9007(96)00376-6]

PACS numbers: 05.45.+b, 02.60.Cb, 05.40.+j

The study of chaos in dynamical systems has attracted the interest of a growing number of scientists during the past decades. In the case of dissipative systems, many of these studies mainly focused on the observable evolution, i.e., the evolution on a chaotic *attractor*. However important, attractors are not the only invariant sets that are present in chaotic dynamical systems [1]. Moreover, invariant sets that are not attracting also play a major role in determining the observable dynamics, in such a way that their structure can even be obtained from real experiments (see, e.g., [2]). In particular, chaotic attractors contain an infinite number of unstable (i.e., not attracting) periodic orbits, which constitute a main ingredient for the occurrence of chaos. In fact, they are related to the *sensitivity to initial conditions* displayed by the attractor, a property that is at the heart of the definition of chaos.

One way of characterizing a chaotic evolution is by means of the *Lyapunov exponents*. These numbers quantify the average rate of expansion and contraction along the different directions in phase space as the system evolves. A positive Lyapunov exponent means that there is an expanding direction, and, therefore, sensitivity to initial conditions and chaos. The periodic orbits inside a chaotic set can also be used to calculate Lyapunov exponents [3] and the number of their expanding directions typically determines the number of positive ones. Most attractors studied in the past contained periodic orbits with equal numbers of expanding directions. However, it has been observed that there can be chaotic attractors where periodic orbits with different numbers of expanding directions coexist. [4,5]. But what happens to a typical trajectory on such an attractor that visits the vicinity of all these orbits? One way of characterizing its behavior is by means of the *finite-time* Lyapunov exponents which quantify the rate of expansion and contraction during a finite time span T [6]. In this case, as the trajectory visits regions with different numbers of expanding directions, the finite-time Lyapunov exponents experience fluctuations on which they change sign. Recent studies have shown that these fluctuations are associated to very complicated behaviors. In particular, they affect the "accuracy" of numerically computed chaotic orbits dramatically [7]. This effect is particularly acute when at least one exponent fluctuates about zero, in which case the validity of the numerical simulations is highly unreliable. Since many studies of chaotic dynamics rely on computer generated data, the validity of the simulations is an extremely important issue. Fluctuating Lyapunov exponents are also associated to other dramatic behaviors, such as the occurrence of riddled basins, blowout bifurcations, and on-off intermittency [8]. It is of interest then to understand the mechanisms by which fluctuating Lyapunov exponents may arise.

We show in this Letter under which circumstances periodic orbits with different numbers of expanding (i.e., unstable) directions and trajectories that repeatedly come close to both of them may coexist inside the same attractor in a *robust* way. Typical trajectories in this attractor will display fluctuating Lyapunov exponents. This behavior, which persists under perturbations of the system, is guaranteed by the existence of a nonattracting chaotic set that has some of the properties of strange attractors. Typical examples of nonattracting chaotic sets in maps include *repellers* [9] (p. 269) and *saddles*, such as those that appear in connection with chaotic transients and fractal basin boundaries [10]. The invariant sets that we present in this Letter are saddles that, unlike the latter ones, are smooth along some unstable directions. They may exist for maps of more than two dimensions (2D) and we expect them to appear as frequently as strange attractors.

We now address the question of how a strange attractor can contain periodic orbits with different numbers of expanding directions and typical trajectories that repeatedly come close to all of them. In order to understand this, we first consider the simplest case of an invertible 3D map for which a trajectory visits the neighborhood of two fixed points, p_1 and p_2. Let us assume that p_1 has two expanding and one contracting direction while the situation is reversed for p_2. Associated with these directions, which determine the local behavior near the fixed points,

 0031-9007/96/76(23)/4348(4)$10.00

there are global stable and unstable manifolds, $W^s(p)$ and $W^u(p)$, which are the set of points that tend to the fixed point p under forward and backward iterations of the map, respectively. Therefore, we may have a trajectory that visits the neighborhoods of both fixed points if the stable manifold of p_1, $W^s(p_1)$, intersects the unstable manifold of p_2, $W^u(p_2)$, and vice versa, as shown schematically in Fig. 1(a). We now ask how robust is the situation; that is to say, what happens if we slightly perturb the map? Since $W^s(p_2)$ and $W^u(p_1)$ are 2D manifolds in a 3D phase space, they typically intersect *transversely* at curves such as C. A small perturbation will simply deform C but will not destroy it. Think, for example, of two planes in R^3 intersecting along a line: they will still intersect if we slightly change their positions. However, the intersection between $W^s(p_2)$ and $W^u(p_1)$ is not robust: it can be destroyed by an arbitrarily small perturbation. Think, for example, of two lines intersecting at a point in R^3; a slight change in their positions will typically split them apart. So, we see in this example that although we may have an orbit that visits the vicinity of two fixed points with different numbers of expanding directions, the situation is not robust and will occur only for very particular cases. We may think that going to higher dimensions will solve the problem. Unfortunately, the same situation arises in all dimensions. In order to show this, let us consider the case of an nD map with two fixed points, p_1 and p_2. Let us assume that the stable manifolds are of dimension $n^s(p_1)$ and $n^s(p_2)$ and the unstable ones of dimension $n^u(p_1) = n - n^s(p_1)$ and $n^u(p_2) = n - n^s(p_2)$. Thus, $W^s(p_1) \cap W^u(p_2)$ will *typically* be of dimension $n^s(p_1) + n^u(p_2) - n = n^s(p_1) - n^s(p_2)$ and $W^u(p_1) \cap W^s(p_2)$ of dimension $n^s(p_2) - n^s(p_1)$. Each intersection will be transverse (and thus, "robust") only if it does not have a negative dimension. However, this cannot be true for both intersections simultaneously if $n^s(p_1) \neq n^s(p_2)$. Replacing p_1 and p_2 by a larger but finite number of periodic orbits does not solve the problem, either. Thus, the situation depicted in Fig. 1(a) seems to be rather unusual. We then ask whether it is possible to have a robust situation where periodic orbits with different numbers of expanding directions belong to the same attractor and there are trajectories that repeatedly come close to all of them.

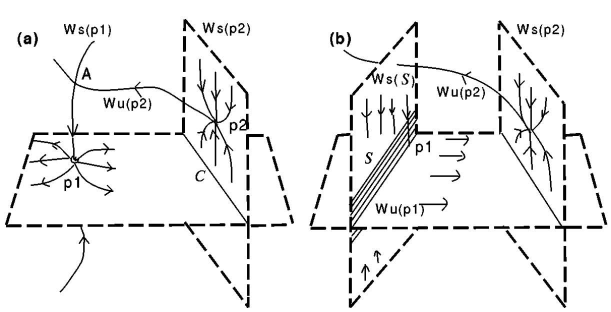

FIG. 1. (a) p_1 and p_2 are fixed points with different numbers of expanding directions and intersecting stable, W^s, and unstable, W^u, manifolds. A slight perturbation destroys the intersection at A. (b) S is a nonattracting chaotic set with a dense set of 2D unstable periodic orbits, like p_1. $W^u(p_2)$ intersects $W^s(S)$ and $W^u(p_1)$ intersects $W^s(p_2)$. The intersections are not destroyed by any small perturbation. Note that only a piece of the manifolds is shown in (a) and (b).

In order to give a hint on how to go around this problem, let us recall how the structure is of a typical strange attractor, $\mathcal{A}$, of a 2D dissipative map. The attractor contains a dense set of unstable periodic orbits, each of which has a 1D stable and a 1D unstable manifold. The attractor is smooth along the unstable directions and has a 2D stable set, the *basin of attraction*, which is the set of points that tend to the attractor by forward iteration of the map. Thus, strange attractors have stable sets of larger dimension than the stable manifolds of its periodic points, p, and such that the difference between $n^s(\mathcal{A})$ and $n^s(p)$ is an integer. This has the important consequence that, in order to approach the vicinity of any of the periodic points contained in the attractor, a trajectory just needs to intersect a 2D object, the basin, which is "easier" than intersecting the 1D stable manifold itself.

Typical chaotic saddles of 2D dissipative maps also contain a dense set of periodic orbits with 1D stable and 1D unstable manifolds, but have a Cantor set structure along the unstable direction [10]. Consider, for example, a 2D map with a chaotic set on which periodic orbits are dense. If it is smooth along the unstable manifolds and in general transverse to the stable ones, then it will have a 2D stable set and will therefore be attracting. On the other hand, if it is not smooth along the unstable manifold, it will be a saddle. The same argument extends to the case of nD maps with 1D unstable manifolds. However, in these higher-dimensional cases, nonattracting chaotic sets with more than one expanding direction can have a different structure. In particular, they can be smooth along some of the expanding directions [see Fig. 1(b)]. Consider, for example, the following 3D map:

$$\begin{aligned} x_{n+1} &= \mathrm{sech}(z/d)(1 - z)(a - x_n^2 + by_n) + zcx, \\ y_{n+1} &= x_n, \\ z_{n+1} &= -z_n^2 - 2z_0 z_n, \end{aligned} \tag{1}$$

where a, b, c, d and $\frac{1}{2} < z_0 < 1$ are parameters. This map is not invertible but it is simple enough so that we can analyze it rather easily: (i) The evolution in z is independent of x and y and is given by a quadratic map with an unstable fixed point at $z = 0$ and attractors with $-3 < z < 1$ for all $\frac{1}{2} < z_0 < 1$. (ii) The $z = 0$ plane is invariant under forward iteration of (1), all of its points are unstable along the z direction, and the evolution on it is given by the Hénon map. Thus, if $a = 1.4$ and $b = 0.3$, there is a chaotic saddle in this plane that has the same structure as the chaotic attractor of the Hénon map with one extra unstable direction along z. All the unstable periodic orbits on the saddle have one contracting and two expanding directions. The saddle, like the Hénon attractor, is smooth along the unstable directions contained in the $z = 0$ plane and is typically transverse to the stable ones. Thus, there is a 2D set of initial conditions with $z = 0$ (the basin of attraction of the Hénon attractor) that go to the strange saddle asymptotically in time. This is an example of a strange chaotic saddle on which periodic orbits are dense, whose stable dimension is higher than that of the periodic points it contains and differs from it in at least an integer. If we replace the evolution equation for z in (1) by the pair of equations $z_{n+1} = -z_n^2 - 2z_0 z_n + e w_n$, $w_{n+1} = z_n$, we get an invertible 4D map that has exactly the same strange saddle as (1).

Consider now a 3D map that has a strange saddle of this type, S, with a dense set of 1D stable periodic orbits and a periodic point, p_2, outside, with a 2D stable manifold, as shown in Fig. 1(b). The unstable set of S or of any of its periodic points can easily intersect the stable manifold of p_2 since they are all at least 2D. On the other hand, the 1D unstable manifold of p_2 can easily intersect the stable set of S, which is at least 2D. Furthermore, both intersections can be transverse (i.e., robust) simultaneously. Therefore, if they occur for a certain parameter value they will still occur for parameter values nearby. On the other hand, if this happens, we may have a trajectory that repeatedly comes close to p_2 and to S. Thus, due to the existence of a strange saddle that is smooth along some of its unstable directions, we have a robust situation in which a trajectory visits the vicinity of periodic points with different numbers of expanding directions. Given a strange saddle like S, a *crisis* [11] can occur at which its stable set is intersected by a chaotic attractor. Suppose that before the crisis, most of the periodic points in the attractor and in the saddle have different numbers of unstable directions, and that after the crisis there still is a chaotic attractor that contains the saddle. Then, the finite-time Lyapunov exponents should experience fluctuations after the crisis due to the coexistence of periodic points with different numbers of expanding directions in the attractor. Moreover, since crises are associated to the creation of new intersections between stable and unstable manifolds or sets, the occurrence of such a crisis would also imply that of the mechanism depicted in Fig. 1(b). In fact, the map (1) provides an example. If we take $a = 1.85$, $b = -0.25$, $c = 0.3$, and $d = 0.1$, a crisis of this type occurs at $z_0 = z_0^* \approx 0.839$, at which the strange saddle at $z = 0$ is incorporated into the attractor. In this case all periodic orbits in the saddle have two expanding directions while most of the orbits in the precrisis attractor have only one [12]. After the crisis, the second Lyapunov exponent starts to fluctuate (see Fig. 2 for another example). Using well known properties of the quadratic map, it can be proved that, if (1) has an attractor at $z_0 = z_0^*$, it must contain points with $z = 0$ (see, e.g., [9], p. 51). For the chosen parameter values, these points belong to the stable set of the strange saddle. It then follows that there is an attractor that contains this saddle. Using the definitions of stable and unstable sets for noninvertible 1D maps [9] (p. 122) we can also show that the mechanism of Fig. 1(b) occurs in this case with the stable and unstable manifolds replaced by stable and unstable sets.

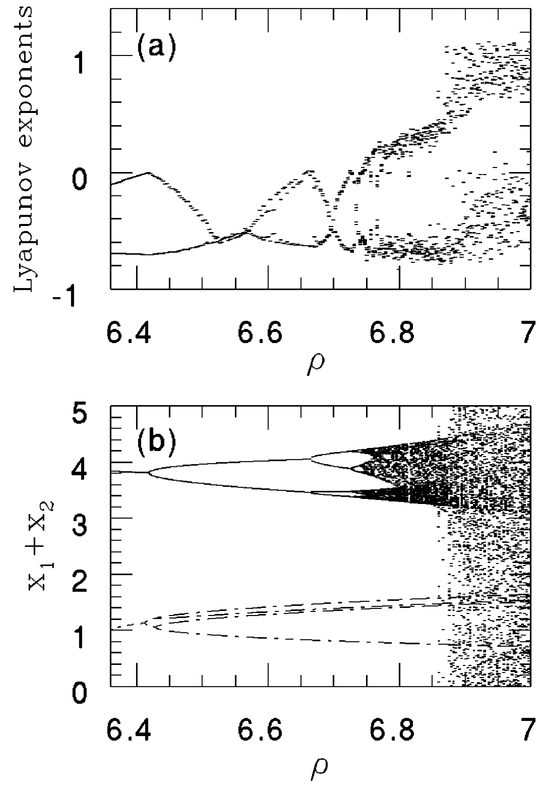

FIG. 2. Two views of the crisis that occurs for the double-rotor map (Ref. [5]) at $\rho \approx 6.86$. (a) Distribution of the first two finite-time Lyapunov exponents as a function of ρ. (b) Bifurcation diagram of the attractors (solid line) and of some 1D unstable periodic orbits (dashed line) that bifurcate by period doubling and become 2D unstable (dash-dotted line).

We now address the question of how often a situation like the one depicted in Fig. 1(b) may be encountered. For this purpose, we first look at the mechanisms by which one such strange saddle can arise in a typical nD dissipative map. To get some insight, we recall how a strange attractor with one positive Lyapunov exponent may appear. A possible scenario involves a sequence of bifurcations as a parameter is varied. At each bifurcation, a stable periodic orbit (i.e., with an nD stable manifold) becomes 1D unstable and gives birth to a new stable orbit which becomes 1D unstable at the next bifurcation and so on, as in Fig. 2(b). When the strange attractor appears, it contains some of these 1D unstable manifolds and is smooth along them. Moreover, the attractor "inherits" the nD stable set of the periodic orbit that was its seed: its nD basin of attraction contains a smooth deformation of what was the stable manifold of the periodic orbit from which the bifurcations started. Consider a completely similar situation, but where we start off a 1D unstable (i.e., not attracting) orbit with an $(n-1)$D stable set. At each bifurcation point, a 1D unstable orbit will become 2D unstable and give birth to a 1D unstable one, as in Fig. 2(b). At the end of this process we expect to have a nonattracting chaotic set, which contains some of these 2D unstable periodic orbits, that is smooth along the unstable direction that was stable before the bifurcations and that has an $(n-1)$D stable set inherited from the "primordial" 1D unstable orbit. For example, this is what happens to the invariant set at $z = 0$ of (1) as we vary ρ. Furthermore, we have found these bifurcations in the double-rotor map [5] [see Fig. 2(b)]. This 4D map is the return map of a system describing the evolution of two connected rods moving on a plane under the effect of δ kicks and damping. For certain parameter values, this map has periodic orbits with different numbers of expanding directions and fluctuating Lyapunov exponents. We have thoroughly analyzed the validity of its numerically computed chaotic trajectories and found that they are highly unreliable when one Lyapunov exponent fluctuates about zero [7].

Is it possible that the mechanism of Fig. 1(b) occurs in this map? In order to answer this, we have studied what happens as the strength of the forcing, ρ, is increased while the moments of inertia, damping coefficients, and time between kicks are kept fixed at the same values as in [5]. We have found a crisis at $\rho \approx 6.86$, after which the Lyapunov exponents start to fluctuate wildly [see Fig. 2(a)]. We have also estimated how far there is a true orbit from the numerically computed ones before and after the crisis, finding that the average of this *shadowing distance* [7] increases by an order of magnitude after the crisis. As shown in Fig. 2(a) the fluctuations persist for a whole interval of parameter values, implying the occurrence of a robust mechanism. Before the crisis there are strange attractors with one positive Lyapunov exponent. Simultaneously, there is a 1D unstable periodic orbit in the boundary of the basin of attraction that becomes 2D unstable via a period-doubling bifurcation. As ρ is increased, it gives rise to a cascade of period-doubling bifurcations at which 1D unstable orbits become 2D unstable [see Fig. 2(b)]. Searching for periodic orbits we could follow only a finite number of their bifurcations. However, using a different technique, we have recently found that, after the bifurcations, there is a strange saddle containing some of these orbits, with a dense set of 2D unstable periodic orbits, two positive Lyapunov exponents, and a 3D stable set [13]. Now, if the unstable manifold of some of the periodic orbits in the precrisis attractor intersects this stable set, the strange saddle will be incorporated into the attractor. In fact, we believe that this is what occurs at the crisis, as shown in Fig. 2(b), where we have plotted the bifurcation diagram of the attractor and of some of the unstable periodic orbits we mentioned before. We have verified that all these orbits belong to the attractor after the crisis. The persistence of the situation together with the facts that all the periodic orbits are brought into the attractor simultaneously and that they are 2D stable while the attractor's dimension is slightly less than 1.7 support the idea that a whole invariant set containing these orbits is incorporated into the attractor at the crisis.

It was suggested in [7] that chaotic attractors containing periodic orbits with unstable manifolds of different dimensions should be encountered very frequently in high-dimensional systems. In this Letter we provide an explanation of how this may arise and why we may expect it to occur so often. The situation of a trajectory, not necessarily in an attractor, that came close to periodic orbits with different numbers of expanding directions was also analyzed in [4]. The construction involved invariant smooth manifolds instead of strange sets as in this case. We believe our situation should be more common than this one in typical dissipative maps. In particular, our two examples suggest that the strange chaotic saddles that we need can be created in the same way as strange attractors. Thus, we think they should appear as frequently as strange attractors in maps of more than two dimensions.

I acknowledge useful conversations with J. A. Yorke, T. Sauer, C. Grebogi, and E. Ott. This work was supported by the University of Buenos Aires, CONICET, and Fundación Antorchas.

*Electronic address: silvina@iafe.uba.ar

[1] In this Letter we describe the evolution with maps, which can be obtained from the corresponding flows via a Poincaré surface of section. A subset of phase space is invariant if it is mapped onto itself as the system evolves forward or backwards in time.

[2] D. P. Lathrop and E. Kostelich, Phys. Rev. A **40**, 4028 (1989); W. L. Ditto *et al.,* Phys. Rev. Lett. **65**, 3211 (1990); G. B. Mindlin *et al.,* J. Nonlinear Sci. **1**, 147 (1991); I. M. Jánosi *et al.,* Phys. Rev. Lett. **73**, 529 (1994).

[3] C. Grebogi *et al.,* Phys. Rev. A **36**, 3522 (1987); C. Grebogi *et al.,* Phys. Rev. A **37**, 1711 (1988).

[4] R. Abraham and S. Smale, Proc. Symp. Pure Math (AMS) **14**, 5 (1970); M. W. Hirsch *et al., Invariant Manifolds* (Springer-Verlag, New York, 1977), p. 141.

[5] R. Romeiras *et al.,* Physica (Amsterdam) **58D**, 165 (1992).

[6] The "usual" Lyapunov exponents are recovered in the limit $T \rightarrow \infty$.

[7] S. Dawson *et al.,* Phys. Rev. Lett. **73**, 1927 (1994).

[8] E. Ott *et al.,* Phys. Rev. Lett. **71**, 4134 (1993); E. Ott and J. Sommerer, Phys. Lett. A **188**, 39 (1994).

[9] R. Devaney, *An Introduction to Chaotic Dynamical Systems* (Addison-Wesley, New York, 1993).

[10] E. Ott, *Chaos in Dynamical Systems* (Cambridge University Press, Cambridge, England, 1993), p. 151.

[11] C. Grebogi *et al.,* Physica (Amsterdam) **7D**, 181 (1983); C. Grebogi *et al.,* Phys. Rev. A **36**, 5365 (1987).

[12] Because of the sech term, the map is contracting along x and y for $-3 < z \ll -d$, $d \ll z < 1$ provided that c is small enough as in this case.

[13] P. Moresco and S. P. Dawson (to be published).