Dynamical Entropies Applied to Stochastic Resonance

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We calculate dynamical entropies from experimental data produced by a Schmitt trigger subjected to noise and a periodic forcing. Both input and output signals are converted to binary sequences. Conditional and Kullback entropies exhibit extrema for certain values of noise intensity. These extrema can be interpreted and will be related to the synchronization effect of switching events induced by external periodic bias. [S0031-9007(96)00380-8]

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Small periodic signals forcing nonlinear systems can sometimes be amplified by addition of a stochastic force to the signal. This effect, called *stochastic resonance* (SR) [1,2], in the past has attracted much attention in the field of nonlinear stochastic dynamics. Besides a great deal of theoretical studies, of numerical simulations, and of analog simulations, SR has been discussed for a variety of applications. The investigations of this phenomenon in biological systems focusing on sensory neuron activity are of great interest [1–3].

Usually SR is characterized by the existence of a maximum in the signal-to-noise ratio (SNR) vs noise intensity relation. Another measure for the SR, the amplification, was introduced in [4] as a ratio of the magnitudes of ensemble averaged response and the input signal. An alternative is the residence-time probability distribution introduced in [5] yielding a bona fide resonance of the forced system [6]. In nonlinear regimes, where the amplitude of the signal is sufficiently strong, synchronizationlike phenomena can be observed [7]. In such regimes the SNR possesses two maxima [8]: the first maximum corresponds to the strong synchronization between hopping events and periodic force while the second one refers to a decrease of the noise background.

As was pointed out by Moss [9] already in 1989 one may associate the switching events in a stochastic bistable and threshold system with an information flow through the system [10]. Indeed, one of the major motivations for SR research is the intuitive idea of gaining information when passing a small (noise affected) signal through an optimally noise tuned bistable or threshold system, i.e., signal detection and transformation [11,12].

In the present work we apply the concepts of dynamical entropies [13] and of the Kullback entropy [14]. Generally, their values give the average amount of information gained after observing (or needed to predict) the outcome of an experiment or measurement process, e.g., a signal, possibly with respect to some preknowledge or some preassumptions, e.g., the unbiased guess.

We will show that SR is accompanied by an extremal behavior of these quantities when sweeping noise intensity over the resonance region. We will prove that an increase of applied noise at first leads to an increase of temporal order in the output sequence. Simultaneously, the predictability of the output sequence will improve in the resonance region. From the Kullback-entropy analysis we will learn that for a certain value of noise intensity the input and output distributions of binary subsequences maximally match. Moreover, input and output subsequences registered at equal times exhibit maximum correlation. Finally, we relate these results to the synchronization mechanism occurring in the system.

The experimental system under investigation is an electronic two-state device: the Schmitt trigger [9,15]. The Schmitt trigger was subjected to a periodic signal and to noise with a cutoff frequency $f_c = 100$ kHz. The system was driven with a sinusoidal force with frequency $f_0 = 100$ Hz. In all experiments the amplitude A of the periodic signal was sufficiently small to prevent switchings of the trigger in the absence of noise: $A < \Delta U$, where $\Delta U = 150$ mV was the threshold of the Schmitt trigger. The noise intensity D was varied in the range between 25 and 115 mV. However, in order to get synchronization effects more pronounced the amplitude of periodic bias was strong enough to induce the existence of a synchronization region in which the mean switching frequency coincides with the frequency of periodic signal (see, e.g., Fig. 1 in [7]). For the value of amplitude of periodic bias $A = 100$ mV the onset of the synchronization region corresponds to the noise intensity $D = 40$ mV and the SNR takes its first maximum at $D \approx 60$ mV. In the region $40 \le D \le 80$ mV the mean switching frequency practically (within the limits of experimental accuracy) equals the frequency of periodic excitation $f_0 = 100$ Hz. The second maximum of the SNR occurs at $D \approx 120$ mV which is, however, out of the synchronization region and as shown in [8] has nothing to do with the effect of stochastic resonance itself.

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In our discussion we always regard as the input signal the periodic signal plus noise. The continuous input and output signals were transformed to discrete data series by stroboscopic observation; the time window τ was chosen to be approximately a twelfth of the period. Finally, all these time series were mapped onto binary symbol sequences accounting for the bistability. Both these binary input and output sequences were the fundamental data for our information theoretical analysis.

For each of the selected noise intensities and for both input and output we collected an ensemble of 20 binary sequences each of length 15 000 symbols. All of the following calculations were accompanied by ensemble statistics yielding mean values and standard deviations. The standard deviations generally were much smaller than mean values.

Let us now introduce the measures we used: Let $\mathbf{i_n} := i_1, \ldots, i_n$ be a binary subsequence. The stationary probability to observe this subsequence shall be denoted by $p(\mathbf{i_n})$. Then the n-block entropies are defined by

$$H_n = -\sum_{(\mathbf{i_n})\in\{0,1\}^n} p(\mathbf{i_n}) \mathrm{ld} p(\mathbf{i_n}) . \tag{1}$$

The symbol ld denotes the logarithm base 2. The n-block entropy (1) is interpreted as the average information necessary to predict a subsequence $(i_1, \ldots, i_n)$, of length n or, equivalently, as the average information gained after its actual observation.

The conditional entropies h_n are introduced for $n = 1, 2, \ldots$ by

$$h_n = H_{n+1} - H_n \tag{2}$$

$$= \left\langle -\sum_{i_{n+1}} p(i_{n+1} \,|\, \mathbf{i_n}) \mathrm{ld} p(i_{n+1} \,|\, \mathbf{i_n}) \right\rangle_{(\mathbf{i_n})} , \tag{3}$$

where the brackets indicate averaging over the prehistory $p(\mathbf{i_n})$. This definition is supplemented by $h_0 := H_1$. Here, $p(i_{n+1} \,|\, \mathbf{i_n})$ denotes the probability for the symbol i_{n+1} conditioned by the n preceding symbols $\mathbf{i_n}$. The h_n are interpreted as the average information necessary to predict the symbol i_{n+1} (or gained after its observation) given knowledge of $\mathbf{i_n}$. Correlations existing between the symbols of a sequence generally decrease this amount of information with increasing the length n of observed prehistory.

Starting from the binary input (output) sequences we computed the related n-word probability distributions (by simple word counting) for $n = 1, \ldots, 16$ and in the sequel the conditional entropies h_n for $n = 0, \ldots, 15$. Figure 1 depicts the h_n of the output sequences as a function of the noise intensity D. A nonmonotonic structure of the curves becomes visible only for $n > 5$, i.e., after having registered a half period in advance which is plausible. The minimum occurs for $D = 60$ and corresponds to the most ordered structure of the output sequence, i.e., when the sequence maximally reflects the periodic structure.

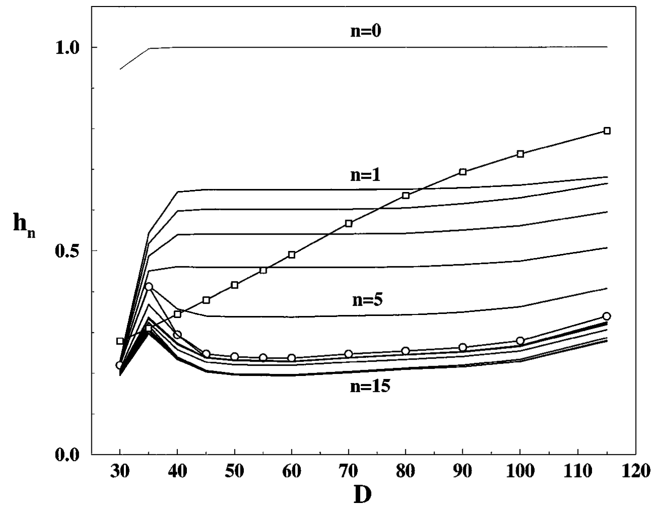

FIG. 1. The conditional entropies h_n [formula (3)] for the output signal (h_6 with circles), $n = 0, \ldots, 15$ being the curve parameters, and h_6 for the input signal (with squares).

A second indication for the period is the decrease of conditional entropies with respect to increasing n (vertical profile): significant jumps occur only for values of n which match multiples of the half period. The decline of conditional entropies for extremely small values of noise intensity, i.e., for $D < 35$, is caused by the fact that the output signal exhibits an intermittentlike character; the system stays in each of the two minima for comparatively long times. An intermittent sequence is a highly ordered structure and because of that is easy to predict [16].

Figure 1 additionally includes a plot of h_6 computed from the input signal. In contrast to the behavior observed for the output h_6, a monotonic increase of the input entropy reflects the constantly growing randomization when increasing the noise intensity. Moreover, one can see that the input sequence rather rapidly gets randomized; i.e., the related conditional entropies approach the maximum value 1 for white noise, while the output sequence more or less preserves the periodic structure. This means even a rather noisy signal bearing only a weak reminiscence of a periodic structure can be efficiently filtered.

The limit entropy, defined by

$$h := \lim_{n\to\infty} h_n \tag{4}$$

is the minimum amount of information necessary for a prediction of the next symbol even when accounting for all correlations, i.e., being informed about the complete prehistory. For SR the limit entropy can be related to the residence time distribution [5]. This connection rests on the assumption that a binary sample sequence equivalently can be constructed by independently choosing subsequent residence times according to the residence time distribution and concatenating alternating laps of 0's and 1's. Then the following formula applies [17]:

$$h = \frac{H[p_{\mathrm{res}}]}{\langle t_{\mathrm{res}} \rangle}, \tag{5}$$

where $H[p_{\mathrm{res}}]$ denotes the Shannon entropy of the residence time distribution and $\langle t_{\mathrm{res}} \rangle$ the average residence time. Figure 2 depicts the result of our computation for the system under investigation. Clearly, this plot qualitatively is in agreement with Fig. 1. Therefore we can realize that the minimum of the limit entropy corresponding to the most ordered structure of the output binary sequence refers to the first maximum of SNR.

Next we address the application of Kullback entropy. Generally the Kullback entropy $K[p^0, p]$ is defined by

$$K[p^0, p] \coloneqq \sum_i p_i \log \frac{p_i}{p_i^0}. \tag{6}$$

Here, p^0 (p) denote an initial (final) probability distribution both with respect to the same set of events. Then the average information gained when replacing initial p^0 by final p, perhaps due to some measurement, is given by $K[p^0, p]$. This quantity establishes a measure for the distance between the two distributions p^0 and p. $K[p^0, p]$ is always greater than or equal to zero and it vanishes if and only if p^0 and p are identical.

We consider the distributions of subsequences of length n related to the binary input (output) sequences; i.e., we identify $p_i^0 \coloneqq p_n^{\mathrm{in}}(\mathbf{i_n})$ and $p_i \coloneqq p_n^{\mathrm{out}}(\mathbf{i_n})$. The result of our computation is shown in Fig. 3; n ranges from 1 to 8. Common to all curves is a relatively pronounced minimum for $D = 40$. This indicates that for this value of noise intensity both distributions maximally match. It does not mean that the output sequence maximally reflects the periodic structure; this happens for $D = 60$, hence, for larger noise intensity. For very small noise intensities the input sequence is closest to the periodic structure whereas

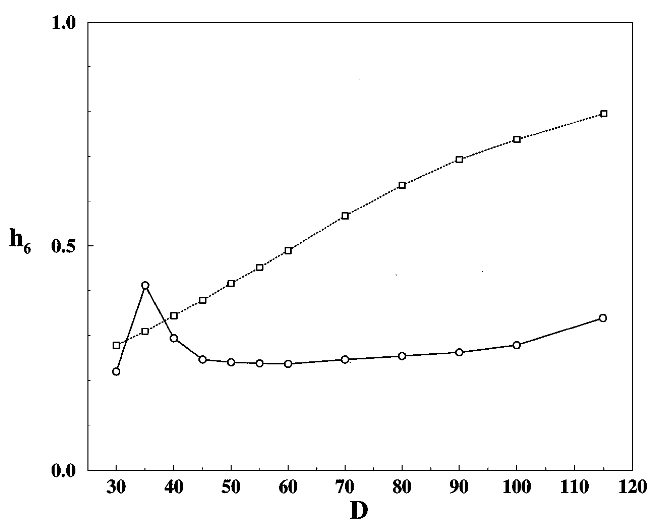

FIG. 2. An approximation to the source entropy h [formula (5)] of binary output (full lines with points) and input sequences (dotted lines with squares) as a function of the noise intensity D.

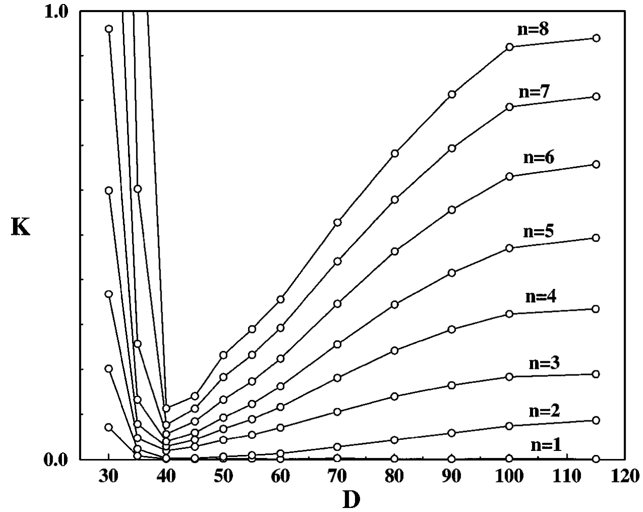

FIG. 3. The Kullback information $K[p_n^{\mathrm{in}}, p_n^{\mathrm{out}}]$ (see text) as a function of the noise intensity D.

the output sequence is intermittentlike. Accordingly both distributions are vastly different. Increasing the noise intensity towards the resonance region the periodic characteristics of the input sequence slightly get blurred. But now the output signal acquires more and more periodic structure. Hence, both distributions converge. Beyond the minimum value $D = 40$ the output signal continues approaching its best periodic shape. But now the input signal gets increasingly randomized. This results in a divergence of both distributions even before the resonance region around $D = 60$ is reached. We note that the value of the noise intensity $D = 40$ at which the Kullback entropy takes its minimum exactly corresponds to the onset of the region of the synchronization (see Fig. 1 in [7]).

It has to be mentioned that $K[p_n^{\mathrm{in}}, p_n^{\mathrm{out}}]$ measures only the distance between the *statistics* extracted from the input (output) signal but makes no statement about correlations existing between segments of the input (output) signal. In order to do that again we employ the Kullback measure but use the following distributions: $p_i^0 \coloneqq p_n^{\mathrm{in}}(\mathbf{i_n}) p_n^{\mathrm{out}}(\mathbf{o_n})$ and $p_i \coloneqq p_{n,n}^{\mathrm{in,out}}(\mathbf{i_n}, \mathbf{o_n})$ where $\mathbf{o_n} = (o_1, \ldots o_n)$.

This Kullback entropy measures the average information gained when replacing the assumption of uncorrelated input and output signals by the observation of correlations. The stronger both signal segments (sampled at equal times) are correlated, the larger will be its value. This quantity can be expressed by two entropies [18]

$$K[p_n^{\mathrm{in}} p_n^{\mathrm{out}}, p_{n,n}^{\mathrm{in,out}}] = H_n^{\mathrm{out}} - H_n^{\mathrm{out}\,|\,\mathrm{in}} \le H_n^{\mathrm{out}}. \tag{7}$$

H_n^{out} is the standard n-block entropy [see (1)] related to the output sequences. $H_n^{\mathrm{out}\,|\,\mathrm{in}}$ is a generalized conditional entropy [analogous to (3)] now based on the probability of observing in the output sequence the n word $\mathbf{o_n}$ conditioned by the n word $\mathbf{i_n}$ at equal time in the input sequence.

Relation (7) can be used to normalize $K[p_n^{\mathrm{in}} \times p_n^{\mathrm{out}}, p_{n,n}^{\mathrm{in,out}}]$

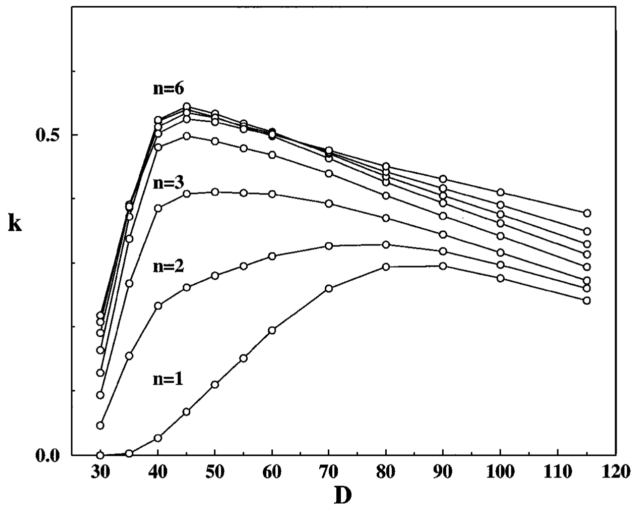

FIG. 4. The normalized Kullback information $k[p_n^{\mathrm{in}} \times p_n^{\mathrm{out}}, p_{n,n}^{\mathrm{in,out}}]$ (see text) as a function of the noise intensity D. The peak around $D = 45$ mV attains a maximum value for the half period $n = 6$.

$$k[p_n^{\mathrm{in}} p_n^{\mathrm{out}}, p_{n,n}^{\mathrm{in,out}}] \coloneqq \frac{K[p_n^{\mathrm{in}} p_n^{\mathrm{out}}, p_{n,n}^{\mathrm{in,out}}]}{H_n^{\mathrm{out}}} \leq 1 \,. \tag{8}$$

Results of applying this many-time correlation measure to our system are displayed in Fig. 4. The most striking observation is a peak around $D = 45$ which becomes most pronounced for word lengths close to the half period of the underlying periodic signal. In this way this plot contains both the information about the period *and* the region of maximum synchronization. Again, this region of maximum synchronization does not coincide with the region of maximum order in the output. Moreover, notice that a constant phase shift between input and output sequences cannot be detected by this measure.

In conclusion, we have studied SR in the Schmitt trigger system by employing the concept of dynamical entropies. We used an experimentally realized system whose input and output signals were converted to binary sequences. The limit entropy attained its minimum and SNR took its maximum for the same value of noise intensity which refers to the most ordered output signal. Two Kullback type measures were employed. First we detected the minimal distance between input and output switching statistics. The second measure was designed to find the state of maximally correlated input and output signals. Both extrema coincided for values of noise intensity corresponding to the onset of synchronization between periodic excitation and switching events. In summary, the measures presented here apply to three different properties controlled by noise: best predictability and best matching of distributions and maximal correlations. In practice, one has to decide which of these properties is the most relevant for signal transmission. The application to other nonlinear stochastic phenomena, e.g., to threshold devices [3,19] or to resonant activation [20], seems to be possible.

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