Lagrangian Formalism for the Rayleigh-Taylor Instability

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A new formulation of the Lagrangian equations for the evolution of the Rayleigh-Taylor instability in inviscid incompressible fluids is presented. A set of exact coupled-mode equations which govern the evolution of the velocity field and the nonlinear motion of the surface is derived. Unlike the traditional Eulerian mode expansion, which requires an infinitely growing number of modes at the nonlinear stage, the present expansion converges very rapidly. The use of the formalism for analyzing the nonlinear stage of the instability is demonstrated by analytical and numerical solutions. [S0031-9007(96)00298-0]

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The Rayleigh-Taylor (RT) and Richtmyer-Meshkov (RM) instabilities occur in many fields of physics: astrophysics, nuclear collisions, atmospheric physics, geophysics, and various configurations of magnetically and inertially confined fusion experiments [1].

The instability occurs when a light fluid supports a heavier one against gravity, or pushes it in a constant acceleration (RT instability [2]), or after a shock has passed through the interface between two fluids (RM instability [3]). Initially, random perturbations at the interface grow exponentially in time. In the nonlinear stage of the instability, the interface is strongly distorted. Round "bubbles" of light fluid enter the heavy fluid and narrow "spikes" of heavy fluid penetrate the lighter one. Eventually the "bubbles-spikes" structure breaks down and a turbulent mixing of the two fluids occurs [4,5]. The present work concentrates on the linear and nonlinear stages of the instability before turbulence takes over.

Linear [2,3,6] and nonlinear [7,8] theories, as well as analyses of full scale numerical simulations [5] and experiments [9], rely on the expansion of the interface and the velocity field in Fourier modes. However, it is well known that this expansion converges only at the very early stage when amplitudes of the modes are smaller than 10% of the wavelength [8,10].

Kull [10] attempted to solve, numerically, the nonlinear fluid equations by expanding the velocity potential and the interface in terms of Fourier modes. He has found that, during time evolution, the convergence of the series becomes increasingly worse until the "Fourier ansatz ceases to be an appropriate representation." The divergence of the expansion occurs as modes amplitudes approach 10% of their wavelengths.

In perturbation expansions [7,8], the nonlinear coupled mode equations are solved by an iterative process starting with the linear solution. Also the perturbation expansion converges only at the very beginning of the nonlinear stage when mode amplitudes are smaller than 10% of the wavelengths [8].

The purpose of the present work was to find an alternative theoretical framework and mode expansion which do not suffer from the shortcomings described above, and can describe also the nonlinear stage of the instability. In the rest of the Letter I will show that Lagrangian formulation of the problem naturally suggests the appropriate mode expansion, and that the resulting mode-coupling equations do not lead to the divergence problem encountered in the Eulerian formulation [7,8,10]. In simple cases the solutions may be represented by a perturbation expansion which converges very rapidly. More complicated problems are treated by a direct numerical solution of the mode coupling equations. The method may be viewed as an extension of Ott's treatment of RT instability in a thin layer [11,12].

Lagrangian formulation of the RT instability.—I shall start with formulation of the Lagrangian equations for the evolution of RT and RM instabilities at the free surface of inviscid incompressible fluid. Surprisingly, such a formulation does not exist, explicitly, in the literature. Consider an incompressible fluid which is supported by a massless fluid. Denote the initial position of a fluid element by $\vec{r}_0$. At later times, the location of the element depends on $\vec{r}_0$. Denote the x, y, and z components of $\vec{r}_0$ by ξ, η, and α. For incompressible fluid, the Jacobian of the transformation from ξ, η, and α to x, y, and z is constant, i.e.,

$$(\vec{r}_\xi \times \vec{r}_\eta) \cdot \vec{r}_a = \text{const}. \tag{1}$$

(Derivatives of $\vec{r}$ with respect to ξ, η, α, and t are denoted by subscripts.) For irrotational motion, the matrix $\vec{\nabla}\vec{v}$ (where $\vec{v}$ is the Eulerian velocity) is symmetric. Consequently, the relation $\vec{r}_\xi \cdot \vec{\nabla}\vec{v} \cdot \vec{r}_\eta - \vec{r}_\eta \cdot \vec{\nabla}\vec{v} \cdot \vec{r}_\xi = 0$ and similar relations for the ξ,α and η,α elements should hold. Using the chain rule for derivatives, these relations are converted to a set of three differential equations which involve only Lagrangian variables:

$$\vec{r}_{\eta t} \cdot \vec{r}_\alpha - \vec{r}_{\alpha t} \cdot \vec{r}_\eta = 0, \qquad \vec{r}_{\xi t} \cdot \vec{r}_\eta - \vec{r}_{\eta t} \cdot \vec{r}_\xi = 0,$$
$$\vec{r}_{\xi t} \cdot \vec{r}_\alpha - \vec{r}_{\alpha t} \cdot \vec{r}_\xi = 0. \tag{2}$$

[Equations (2) may be generalized to include also the case of nonvanishing vorticity. This is done by using the symmetry of the matrix $\vec{\nabla}(d\vec{v}/dt)$, viscosity may be included as source terms which break the symmetry of the matrix $\nabla(d\vec{v}/dt)$].

The above equations [(1), (2)] should be supplemented by boundary conditions. Assuming periodicity in the ξ, η variables and demanding that the fluid velocity will vanish as $\alpha \to \infty$ supply part of the conditions. We still have to determine the boundary conditions at the fluid surface. At $t = 0$ the surface is at $\vec{r} = (\xi, \eta, 0)$. Its location at later times is denoted by $\vec{R}(\xi, \eta, t)$ [i.e., $\vec{R}(\xi, \eta, t) \equiv \vec{r}(\xi, \eta, 0, t)$]. The equation of motion of a fluid element of mass density ρ which is at the surface is $\rho\vec{R}_{tt} = [\vec{\nabla}P(\vec{r}, t)]_{\vec{r}=\vec{R}(\xi,\eta,t)} + \rho g\hat{z}$. Line integrating this equation along a closed rectangular contour with two sides parallel to the surface ($d\vec{R} = \vec{R}_\xi d\xi$ or $d\vec{R} = \vec{R}_\eta d\eta$), within and out of the fluid, taking into account the continuity of pressure across the surface of the fluid one gets the boundary conditions

$$\vec{R}_\xi \cdot \vec{R}_{tt} = g\vec{R}_\xi \cdot \hat{z}, \qquad \vec{R}_\eta \cdot \vec{R}_{tt} = g\vec{R}_\eta \cdot \hat{z} \tag{3}$$

(g is the gravitational acceleration). Equations (1) and (2) together with the boundary conditions at $\alpha = 0$ [Eqs. (3)] and at $\alpha \to \infty$, and the periodicity conditions in the ξ, η variables, uniquely determine the motion of the fluid.

Coupled mode equations.—In the following I shall limit the treatment to systems in which no motion occurs in the y direction. In this case $\vec{r} = (x, \eta, z)$ and $\vec{r}_\xi = (x_\xi, 0, z_\xi)$.

I assume that the displacement of a fluid element $x(\xi, \alpha, t) - \xi$ and $z(\xi, \alpha, t) - \alpha$ may be expanded in a Fourier series, i.e., $x(\xi, \alpha, t) = \xi + \sum_k X_k(\alpha, t)e^{ik\xi}$ and $z(\xi, \alpha, t) = \alpha + \sum_k Z_k(\alpha, t)e^{ik\xi}$ and use the expansion in Eqs. (1) and (2). The result is as follows:

$$\begin{aligned} \dot{Z}'_k + ik\dot{X}_k &= -\sum_{k'} ik'[\dot{X}_{k'}Z'_{k-k'} + X_{k'}\dot{Z}'_{k-k'} \\ &\qquad - \dot{Z}'_k X'_{k-k'} - Z_{k'}\dot{X}'_{k-k'}], \\ \dot{X}'_k - ik\dot{Z}_k &= \sum_{k'} ik'[-\dot{Z}_{k'}Z'_{k-k'} + Z_{k'}\dot{Z}'_{k-k'} \\ &\qquad - \dot{X}_{k'}X'_{k-k'} + X_{k'}\dot{X}'_{k-k'}]. \end{aligned} \tag{4}$$

Similarly, Eq. (3) yields

$$\begin{aligned} \ddot{X}_k(0,t) - gikZ_k(0,t) &= -\sum_{k'} i(k - k')X_{k-k'}(0,t)\ddot{X}_{k'}(0,t) \\ &\quad - \sum_{k'} i(k - k')Z_{k-k'}(0,t)\ddot{Z}_{k'}(0,t). \end{aligned} \tag{5}$$

In the above equations the derivatives of Z_k and X_k with respect to t are denoted by a dot and with respect to α by a prime. One has to solve these equations with the requirement that, as $\alpha \to \infty$, the velocity field vanishes, i.e., $\lim_{\alpha\to\infty} \dot{X}_k(\alpha, t) = \lim_{\alpha\to\infty} \dot{Z}_k(\alpha, t) = 0$. Next, I will present solutions of these equations in the linear and nonlinear regimes.

Analytical solutions.—The solution of Eq. (4), for $X_0(\alpha, t)$ and $Z_0(\alpha, t)$ is $\dot{X}_0(\alpha, t) = 0$ and $Z_0(\alpha, t) = -\sum_{k'} 2ik'X_{k'}(\alpha, t)Z_{-k'}(\alpha, t)$. An exact solution in the case $k \neq 0$ was not obtained, however, linearizing Eqs. (4) one can immediately see that a solution with vanishing $\dot{X}_k, \dot{Z}_k$ at $\alpha \to \infty$ is possible only if at $\alpha = 0$ (i.e., at the interface) $\dot{X}_k$ and $\dot{Z}_k$ are related by $\dot{Z}_k(0, t) = i(k/|k|)\dot{X}_k(0, t)$. Using this result yields the lowest-order nonlinear approximation for the solution of Eqs. (4) and (5). In this approximation, Eq. (5) is reduced to a set of nonlinear ordinary differential equations of the form

$$\ddot{X}_k(0,t) - \sum_{k'} \Gamma^2_{k,k'}X_{k'}(0,t) = 0, \tag{6}$$

where $\Gamma_{k,k'}$ depends on the amplitudes X_k and velocities $\dot{X}_k$ of all the modes in the system.

Note that Eq. (6) has the same form as the standard linear equation for RT instability [6], but with the linear growth rate $\gamma^2 = kg$, replaced by an effective nonlinear Hermitian "growth matrix" $\Gamma^2_{k,k'}$ which couples between modes and depends on the amplitudes and velocities of all the modes in the system. The explicit form of the matrix Γ^2 is $\Gamma^2 = A^{-1}B$ where the elements of the matrices A and B are

$$\begin{aligned} A_{k,k'} &= \left[\delta(k,k') - i(k - k')\left(1 - \frac{(k - k')}{|(k - k')|}\frac{k'}{|k'|}\right)\right. \\ &\qquad \left. \times X_{k-k'} + 2|k|\,|k'|X_kX_{-k'}\right], \end{aligned} \tag{7}$$

$$B_{k,k'} = \delta(k - k')\left[g|k| - |k|\sum_{k''}|k''|(2\dot{X}_{k''}\dot{X}_{-k''})\right]. \tag{8}$$

In systems in which only the lowest N harmonics of the basic mode ($k = 2\pi/L$) have nonvanishing amplitudes X_k, the matrix A^{-1} (and therefore also $\Gamma^2 = A^{-1}B$) does not couple these N modes to higher harmonics. Namely, Eq. (6) does not allow generation of new modes in the system. For example an initial sinusoidal perturbation with an amplitude x_0 and velocity v_0 and a wave number k will evolve as a single Lagrangian mode, i.e., $x(\xi, 0, t) = \xi + 2X_k(0, t)\cos(k\xi)$ and $z(\xi, 0, t) = -2iZ_k(0, t)\sin(k\xi) - 2ikX_k(0, t)Z_k(0, t)$. A single Lagrangian mode $e^{ik\xi}$ is comprised of many Eulerian modes e^{iqx}, each of which has the amplitude $a_q(k, t) = \int e^{-ik\xi}e^{iqx}|\partial x/\partial\xi|d\xi = \int e^{-ik\xi}e^{iq\xi+2iqX_k(0,t)\cos(k\xi)}|1 + 2kX_k(0, t)\sin(k\xi)|d\xi$. For a small Lagrangian mode amplitude $kX_k \ll 1$, one gets that the Eulerian and Lagrangian modes coincide, $\lim_{kX_k\to 0} a_q(k, t) = \delta(k - q)$; the larger the amplitude of the Lagrangian mode, a larger number of Eulerian modes are present.

Solving Eq. (6), using the relations between X_k and Z_k, substituting in the above formulas for $x(\xi, 0, t), z(\xi, 0, t)$ we can now plot the surface $z(x, t)$ at various times. In Fig. 1(a) I show the evolution of the interface in a case with gravitational acceleration $g = 1$. The system is initiated with a single mode with $k = 1$, $x_0 = 0.005$, and $v_0 = 0$ (line A corresponds to $t = 4.2825$ and B to $t = 4.96$). At late times the surface has a falling down spike at $k\xi = \pi/2$ and a rising bubble with a tip at $k\xi = 3\pi/2$.

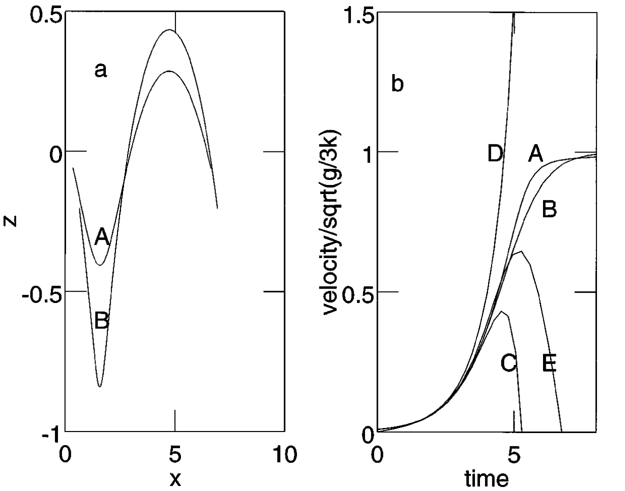

FIG. 1. Analytical results: (a) The surface $z(x, t)$ at various times. (b) Bubble velocity as a function of time.

Higher orders in the perturbation expansion are obtained by iterating the zero order result in Eqs. (4). For example, in the first order, for a single mode, one gets

$$Z_k(0,t) = \frac{1}{3} i \frac{k}{|k|} X_k(0,t) \times \left[\frac{3 - 3kX_k(0,t) + k^2 X_k^2(0,t) + k^3 X_k^3(0,t)}{1 - kX_k(0,t) - k^2 X_k^2(0,t) + k^3 X_k^3(0,t)} \right].$$

[Strictly speaking, in the first order $Z_k(0,t)$ is a polynomial of $X_k(0,t)$; the right hand side of the above relation is a Padde approximant of this polynomial.] As we shall see, this first order result is sufficient for the description of the bubble's velocity. In Fig. 1(b), the bubble velocity is plotted as a function of time (line A). For comparison I have added on this plot also the velocity predicted by the Layzer model [13] (line B) and by second and third order Eulerian perturbation expansion [7] (lines C and D), and the lowest order in the present method, obtained by using the solution of Eq. (6) (line E). Equation (6) describes the bubble's velocity correctly at times $t < 5$. The Layzer model [13] is not capable of describing the whole interface; nevertheless its prediction for the velocity of the bubble's top is known to be accurate. Also the first order in the present approach (line A) describes correctly the velocity of the bubble in the linear stage and at saturation.

Direct numerical solution.—More detailed results for more complicated cases may be obtained by a numerical solution of Eqs. (4) and (5) by discretizing the variables α and t. In contrast to the conventional Eulerian mode expansion which requires an infinitely growing number of modes [8,10], in the present method a bubble-spike structure is described by few modes. Also, since the discretization of the equations in the direction parallel to the interface is obtained by mode expansion and not by a spatial grid, the numerical solution of Eqs. (4) and (5) does not suffer from the problem of mesh distortion which is typical of Lagrangian codes [14].

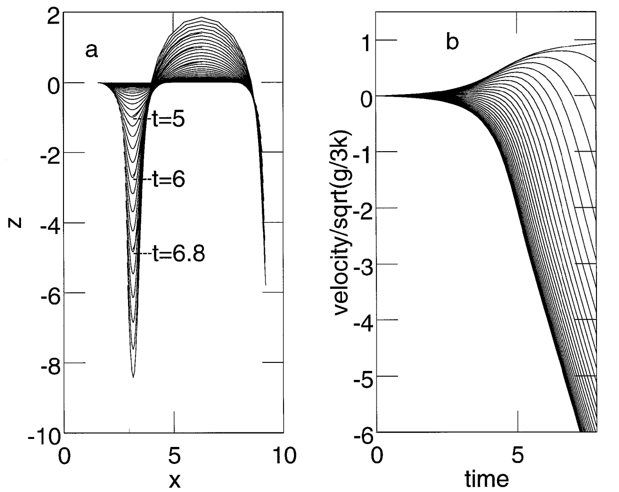

FIG. 2. Numerical results: (a) Evolution of the interface from $t = 0$ to $t = 7.8$ in steps of $\Delta t = 0.2$. (b) The z component of the velocity of fluid elements on the interface [i.e., $\dot{z}(\xi, 0, t)$], (same example as in Fig. 1).

Figures 2 and 3 show the results of the numerical solution for the same example which was treated, analytically, in the previous section. The numerical solution required a grid of 60 points in the α direction and 10 modes. Figure 2(a) shows the evolution of the interface (from $t = 0$ to $t = 7.8$ in steps of $\Delta t = 0.2$). Figure 2(b) shows the z component of the velocity of fluid elements which are on the interface [i.e., $\dot{z}(\xi, 0, t)$], for various values of initial positions (ξ). The upper line approaches the value of $\sqrt{g/3k}$; it corresponds to $\dot{z}(2\pi, 0, t)$, namely, the velocity of a fluid element at the bubble's top. The lower line approaches a constant acceleration g; it corresponds to a fluid element which started at $\xi = \pi$, namely, a fluid element at the spike's tip.

At this point we may compare the results to other numerical simulations. Baker, Meiron, and Orszag [15] have

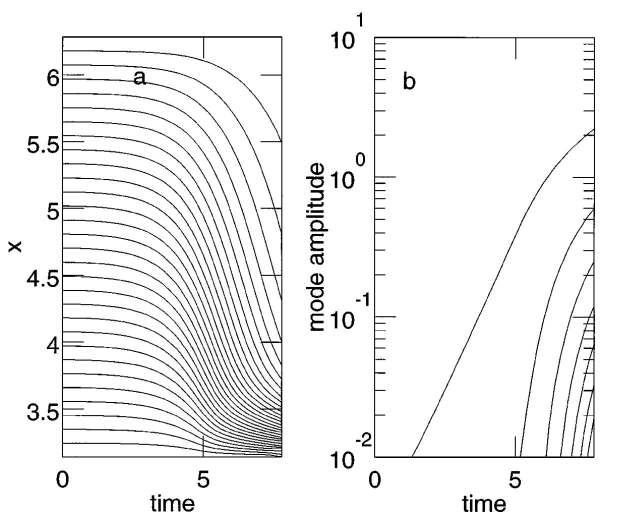

FIG. 3. Numerical results: (a) Positions of fluid elements on the interface $[x(\xi, 0, t)]$ for various values of initial positions (ξ). (b) Mode amplitudes as function of time (same case as in Fig. 1).

used the vortex technique and Menikoff and Zemach [16] have used conformal maps to solve the fluid equations. Their results agree with each other. The example considered in the present work corresponds to the example considered in section (IVA) of Ref. [16]. Their calculation breaks down at $t = 6.8$ due to narrowing and elongation of the spike. In Fig. 2(a) I have added the shape of the interface at times 5, 6, and 6.8 obtained by digitizing Fig. 1 of Ref. [16]; the agreement between the results of the present work and those of Ref. [16] is good. In Refs. [16] and [15] it was found that, before reaching the free fall stage, the spike overshoots to an acceleration larger than g. This phenomenon is explained by a gradient of pressure at the line of symmetry of the spike. Also in the present example, a careful examination of the spike's motion shows that the spike acceleration reaches to the value of $1.2g$ at $t \approx 5$ before stabilizing at the free fall value g.

Ott's solution [11] for the RT instability in a thin layer was limited to times $t < t^*$ when elements of fluid cross, and the interface becomes multivalued. Note that, unlike the case of a thin layer, in the present work, incompressibility [Eq. (1) in r space or equivalently the first of Eqs. (4) in k space] does not allow crossing of fluid elements. Indeed, as is clearly seen in Fig. 2(a), even at saturation, the spike has a round single valued tip. This point is better seen in Fig. 3(a) where the positions of fluid elements on the interface $[x(\xi, 0, t)]$ are plotted as functions of time. Note that fluid elements which are near the spike [namely near the line $x(\xi, 0, t) = \pi$] approach each other very closely but never cross. Figure 3(b) shows the evolution of the Lagrangian modes [i.e., $Z_k(0, t)$ for $k = 1, \ldots, 10$]. At saturation $(t = 8)$ the normalized amplitudes are 1.000, 0.2686, 0.1126, 0.0544, 0.0291, 0.0149, 0.0086, 0.0037, 0.0025, and 0.0002; namely, only the first six modes have significant amplitudes of more than 1% of the first mode amplitude.

It is well known that Lagrangian codes, based on spacial meshes, are limited by mesh distortions [14]. Actually even for a single bubble-spike structure a pure Lagrangian calculation cannot be carried from the linear stage into saturation. Remeshing is necessary in order to overcome unacceptably high truncation errors due to mesh distortions. The results presented here are based on a pure Lagrangian calculation; however, instead of a spacial grid, the discretization in the direction parallel to the interface was obtained by a Fourier series expansion. As a result, instead of following fluid elements which undergo extreme distortions [Fig. 3(a)], one follows the smooth evolution of mode amplitudes [Fig. 3(b)]. This feature allows us to carry a pure Lagrangian calculation without remeshing.

To summarize, in this work, a new theoretical framework for the analysis of RT instability was developed. The main result of this approach is the set of Lagrangian coupled mode equations [Eqs. (5) and (4)] which governs the evolution of the velocity field and of the interface. The analytical solution, for the case where the instability is triggered by a single initial mode, shows that perturbative solutions of these equations converge very rapidly. Thus, unlike the traditional Eulerian mode expansion, it may be used for the analysis of RT instability also in the nonlinear stage. The direct numerical solution of the coupled mode equations demonstrates the advantages of the Lagrangian mode picture also for numerical calculations.

The solutions derived in the present work were limited to two dimensional motion of a single fluid. Note, however, that the basic formulation [Eqs. (1) and (2) together with the boundary conditions (3)] is three dimensional. In cases where no motion occurs in the y direction (i.e., 2D motion), the extension of the formulation to the case of two fluids is simple: one uses two sets of Lagrangian variables, Eqs. (1) and (2) hold within each fluid, and the boundary condition (3) is replaced by the condition that the quantity $(\vec{R}_\xi/|\vec{R}_\xi|) \cdot (\rho \vec{R}_{tt} - \rho g \cdot \hat{z})$ is continuous across the interface between the fluids.

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[1] A sample of references for the occurrence of the RT and RM instabilities in various fields of physics may be found in the introduction of B. A. Remington, S. V. Weber, M. M. Marinak, S. W. Haan, J. D. Kilkenny, R. J. Wallace, and G. Dimonte, Phys. Plasmas **2**, 241 (1994).
[2] Lord Rayleigh, *Scientific Papers* (Cambridge University Press, Cambridge, 1900), Vol. II, p. 200; G. I. Taylor, Proc. R. Soc. London A **201**, 192 (1950).
[3] R. D. Richtmyer, Commun. Pure Appl. Math. **13**, 297 (1960); E. E. Maeshkov, Izv. Akad. Nauk SSR Mekh. Zhidk. Gaza **5**, 151 (1960).
[4] For a review, see D. H. Sharp, Physica (Amsterdam) **12D**, 3 (1984).
[5] D. L. Youngs, Phys. Fluids A **3**, 1312 (1991); D. L. Youngs, Laser Part. Beams **12**, 725 (1994).
[6] S. Chandrasekhar, *Hydrodynamic and Hydrodynamic Stability* (Oxford Univ. Press, Oxford, 1961), Chap. X.
[7] J. W. Jacobs and I. Catton, J. Fluid Mech. **187**, 329 (1988).
[8] S. W. Haan, Phys. Fluids B **3**, 2349 (1991).
[9] G. Dimonte and B. Remington, Phys. Rev. Lett. **70**, 1806 (1993).
[10] H. J. Kull, Phys. Rev. A **33**, 1957 (1986).
[11] Edward Ott, Phys. Rev. Lett. **29**, 1429 (1972).
[12] W. Manheimer, D. Colombant, and E. Ott, Phys. Fluids **27**, 2164 (1984).
[13] David Layzer, Astrophys. J. **122**, 1 (1955).
[14] J. A. Zukas *et al., Impact Dynamics* (John Wiley, New York, 1982), Chap. 10.
[15] G. R. Baker, D. I. Meiron, and S. A. Orszag, Phys. Fluids **23**, 1485 (1980).
[16] R. Menikoff and C. Zemach, J. Comput. Phys. **51**, 28 (1983).