Stokes Phenomenon in Chaotic Systems: Pruning Trees of Complex Paths with Principle of Exponential Dominance

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Stokes phenomenon is investigated in the complex semiclassical theory of chaotic tunneling, and a rule for locating contributing complex paths out of a complicated set of candidates is proposed. The proposal is based upon discovery of a tree structure hidden in entangled tunneling paths: The candidates are ordered along the tree, and the *principle of exponential dominance* (PED) is extended to prune noncontributing paths. This rule enables construction of semiclassical tunneling wave functions systematically. A phenomenon for which PED plays a crucial role is also demonstrated. [S0031-9007(96)00018-X]

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An important and difficult problem inherent in the WKB theory is to construct the global solution by connecting local solutions separated by the classical turning points at which the WKB solution becomes singular. This is called the connection problem. One of the efficient methods to cope with it is using the complex plane along which two WKB solutions separated by a turning point are connected [1–3]. In the complex WKB theory, a central subject is how to deal with the *Stokes phenomenon* which is the discontinuous change of the number of contributing solutions across a certain boundary line, called a *Stokes line,* in the complex plane. More generally, if an asymptotic form of a certain function given by the integral form or an asymptotic solution of an ordinary differential equation is expressed as two exponentials,

$$u(z,k) \approx a_+(z,k)\exp[ikw_+(z)] + a_-(z,k)\exp[ikw_-(z)] \quad [\mathrm{Im}w_+(z) < \mathrm{Im}w_-(z)], \tag{1}$$

where k is some large parameter, then there will be a discontinuous change in the multiplier $a_\pm(z,k)$ over several regions in the complex z plane separated by the Stokes lines [4,5].

The original explanation for the Stokes phenomenon given by Stokes himself is based on the argument taking into account how the full asymptotic series can be resummed around the Stokes line [4]. The most intuitive understanding for it is that the number of steepest descent solutions changes as the integral contour is deformed with the argument of the complex variable z. Recent remarkable progress on the Stokes phenomenon is that this change is not discontinuous as has been believed for a long time, but smooth and, moreover, universal [6].

The complex WKB method plays a crucial role in the description of purely quantum phenomenon [7], like the tunneling effect. However, if the degree of freedom is more than one, dynamics of the system is not in general integrable, which makes the complex semiclassical theory very problematic. Indeed it has been shown that an enormous number of contributing complex orbits are associated with the corresponding complex caustics in the semiclassical theory of chaotic tunneling, where the term "chaotic tunneling" has been introduced by the authors to specify tunneling phenomena with characteristic features which are attributable to the chaotic structures of classical trajectories in the complex regime [8]. Here we encounter a serious problem of how to deal with the Stokes phenomenon appearing in complicated circumstances. One cannot, therefore, skip the problem, although it seemingly plays only a technical role in conventional theories of tunneling in completely integrable systems.

Our aim in the present Letter is to propose a working hypothesis for the treatment of the Stokes phenomenon in chaotic systems by taking quantum maps as typical examples. The precise determination of the Stokes line is beyond the conventional WKB theory. However, on the basis of the arguments taking account of full asymptotic expansion [5,6], there is a possible generalization of how to locate the Stokes lines. Such a rule is called the *principle of exponential dominance* (PED) and is written explicitly as follows: "The subdominant contribution appears (or disappears) when the dominant contribution becomes exponentially maximal as compared with the subdominant one. Alternatively stated, the Stokes line should be located where $\mathrm{Im}[w_-(z) - w_+(z)]$ becomes maximal under the condition $|z| =$ const." We adopt this here as a guiding principle to cope with the Stokes phenomenon in chaotic systems. In fact, using a coherent state path integral method, Adachi showed that the PED works well at least in the very initial stage of wave-packet propagation [9]. The problem is, however, that we have to overcome several difficulties prior to applying the PED to chaotic systems.

To be concrete, we introduce the model system which we use for studying the stokes phenomenon in chaotic

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systems. The model is the kicked rotor described by the Hamiltonian

$$H = H_0(\hat{p}) + V(\hat{\theta}) \sum_n \delta(t - n), \qquad (2)$$

where

$$H_0(p) = \frac{p^2}{2} \frac{p^n}{p^n + p_D^n} + \omega p, \quad V(\theta) = K \sin\theta . \qquad (3)$$

If we set $p_D \neq 0$, $\omega \neq 0$, and $n = 6$, the model is the modified kicked rotor with which we examined an idealized situation of chaotic tunneling [8], while it becomes the standard form of a kicked rotor (standard map) for $p_D = \omega = 0$. In the present Letter we examine both cases.

We investigate the tunneling across the momentum (p) space. The semiclassical wave function in the momentum representation at time step t is given as a sum over classical trajectories whose initial momentum p_0 and final momentum p_t both take real values,

$$\psi(p(t,\theta_0)) = \sum_j A_j(p(t,\theta_0)) \times \exp\left[\frac{i}{\hbar} S_j(p(t,\theta_0)) - i\mu\frac{\pi}{2}\right], \qquad (4)$$

where the index μ denotes the Maslov index. As we fix the initial momentum as $p_0 =$ const, the final momentum is described only as a function of the complex initial angle $\theta_0 = \xi + i\eta (\xi, \eta \in \mathbf{R})$ as $p(t,\theta_0)$. The candidate of classical trajectories contributing to Eq. (4) is decided by the condition $p(t,\theta_0) = p_t$, and thus all the initial angles of the candidate trajectories contributing to Eq. (4) at various p_t form a set $\boldsymbol{M}$ satisfying $\mathrm{Im}p(t,\theta_0) = 0$. A typical example of the set $\boldsymbol{M}$ is shown in Fig. 1 by thick lines. In addition to the set $\boldsymbol{M}$, the contours satisfying $\mathrm{Im}p(t,\theta_0) =$ const are also drawn by thin lines. The set $\boldsymbol{M}$ is composed of a very complicated set of branches with various scales, each of which is disconnected from others and looks like a "petal." If (ξ, η) is moved along one such petal, $\mathrm{Re}p(t,\theta_0)$ ranges from $-\infty$ to ∞, and each petal contributes to the semiclassical wave function at an arbitrary fixed value of p_t. This is the reason why we called each petal the branch. This example is obtained for the modified kicked rotor, but the set $\boldsymbol{M}$ for the standard kicked rotor also has essentially the same structure composed of similar petals.

The main subject we discuss hereafter is how to extract the contributing part out of the complicated set of branches forming $\boldsymbol{M}$ by applying the PED. The key is to note the fact that the branches always appear pairwise although they do not make contact with each other. This fact means that their connections occur via complex caustics, which are indicated by black circles in Fig. 1.

Given a caustic and associated pair of branches we can locate Stokes lines bordering a region to be removed as noncontributing. We first describe how the PED may be applied at the caustic, which is now expressed as a function of initial angle θ_0 and defined by

$$\left.\frac{\partial p(t,\theta_0)}{\partial \theta_0}\right|_{\theta_0=\theta_0^*} = \left.\frac{\partial^2 S_t}{\partial \theta_0^2}\right|_{\theta_0=\theta_0^*} = 0, \qquad (5)$$

where θ_0^* denotes the position of a caustic under consideration. The wave function constructed with the associated pair of branches is given as

$$\psi(p(t,\theta_0)) = A(p(t,\theta_0)) \exp\left(\frac{i}{\hbar} S(p(t,\theta_0))\right) + A(p(t,\theta_0')) \exp\left(\frac{i}{\hbar} S(p(t,\theta_0'))\right), \qquad (6)$$

where θ_0 and its partner θ_0' give the same final momentum p_t, and Maslov indices are omitted for brevity. For these two exponentials extracted locally, we compute steepest ascent lines around θ_0^* for the imaginary part of the difference of two exponents,

$$\delta\mathrm{Im}S(p(t,\theta_0)) \equiv \mathrm{Im}[S(\theta_0) - S(\theta_0')]. \qquad (7)$$

There exist three steepest ascent lines from one caustic, and the region around the caustic is separated into three by these boundary lines. We identify such boundary lines with Stokes lines, just on which the subdominant exponential may switch from contributing to noncontributing or vice versa. One of the three regions bordered by the Stokes lines should be removed as noncontributing. Here, however, arise several difficulties. The origin of these difficulties comes from the fact that the PDE is a local principle applicable only close to the caustics.

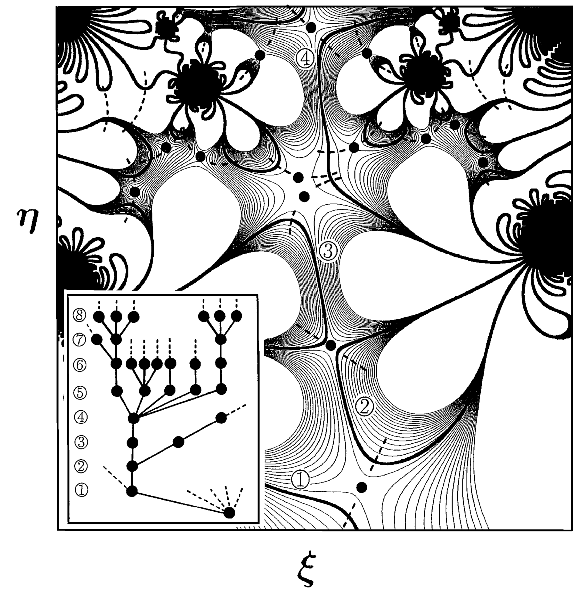

FIG. 1. Typical example of candidate trajectories forming the set $\boldsymbol{M}$ (thick lines) and caustics (black circles). The thin contour lines superposed are $\mathrm{Im}\{p(t, \xi + i\eta)\} =$ const lines. The pair of branches connected by a broken line is the proper pair. The inset is the tree structure hidden behind complicatedly distributed branches in the figure. Tracing the proper pairs, one can recognize the presence of the tree. The number in circles indicates the generation of the branch. A modified kicked rotor ($p_D = 5$, $\omega = 2$, and $n = 6$) is used.

The first difficulty is that the caustic often has a very large imaginary part of p_t, and we have to extend the locally determined Stokes line globally until it intersects with the branch under consideration. Next, from the observation of the behavior of complex classical paths close to the caustic, we cannot in principle identify which of the three regions should be removed, because we have no information on how the integral contour is deformed so as to pass through a saddle point, i.e., a complex classical path. The third difficulty is as follows: Even though we could identify the region to be removed as noncontributing by any local criterion, a branch judged as contributing around a caustic may be included in the region judged as noncontributing around another caustic and vice versa, and a contradiction occurs as to which of the judgments should dominate. Such a contradiction is quite serious in the chaotic tunneling problem in which a large number of branches are connected in a complicated manner.

The first difficulty is rather technical, and it is always possible to extend the Stokes line globally starting from a caustic in a reasonable way [10], but the latter two, which are mutually connected, are an essential problem, which will be discussed below in considerable detail.

To unravel the entanglement among the branches and decide which region to remove, it is necessary to find the order of connection among the branches. Indeed, a surprising fact is that there is a natural order hidden in the complicated set of branches, and it provides a basis to treat such a complicated Stokes problem in chaotic systems.

The basis of the ordering is the "pairing rule," that is, for a given caustic a pair of branches connected via it can always be extracted: considering the line ℓ in the (ξ, η) space satisfying $\mathrm{Re}\,p(t, \xi + i\eta) = \mathrm{Re}\,p(t, \theta_0^*)$, the pair of branches in $\boldsymbol{M}$ with which ℓ extending from θ_0^* in opposite directions first intersects is called the "proper pair" associated with the caustic θ_0^*. Picking up all the caustics and the proper pairs associated with them, all branches are reordered to form a tree structure with the real branch as the common root. In Fig. 1 we show how the branches are paired with respect to caustics, where two branches forming a proper pair are connected by a broken line indicating ℓ in Fig. 1. The inserted diagram is the tree structure constructed from major branches seen in Fig. 1.

It seems possible that descendants of any two branches bifurcated in an earlier stage of the tree structure are connected via a caustic in a later stage. If such is the case, the branches are entangled to form a structure like a rhizome rather than a tree. However, we could not encounter such manifestations empirically [10]. The presence of such a tree structure allows us to define the generation of the branch: Let the real branch be the 0th order generation, and pick up all the partners, each of which forms a proper pair with the 0th order branch. These partners are assigned as the first generation. Repeating the same procedure recursively, we can decide the generation of all the branches starting from the real branch as the common root.

The ordering of complex paths together with the extension hypothesis for Stokes lines provides us with a working hypothesis to extract the contributing part out of the complex branches *systematically:* (1) According to the procedure described above, reorder the branches to form a tree structure, and decide the generation of the branches. (2) Consider a proper pair and decide the extended Stokes lines around the associated caustic. Among the three regions separated by Stokes lines we remove the region containing the unphysical part of the branch of higher-order generation, i.e., the part along which $-\mathrm{Im}S(p_t)$ diverges to the positive infinity (i.e., $|e^{iS/\hbar}| \to \infty$) as $|p| \to \infty$. We show in Fig. 2 the removed regions (shaded regions) together with the Stokes lines. The part of the branches included in the shaded region is removed as noncontributing. (3) All the branches belonging to the higher-order generations of the removed branch part should also be removed.

We have thus a "pruned tree" from which the noncontributing branches are removed. The semiclassical wave function is constructed by summing all the contributions from the branches of the pruned tree according to Eq. (4). The unphysical role of the pruned branches was checked from the observation that the semiclassical wave function deviates explosively from the quantum wave function if the contributions from the pruned branches are included in the semiclassical sum.

Our method described above works surprisingly well as shown in Fig. 3(a): The semiclassical formula (4) reproduces complicated tunneling tails quite well. If the position of some Stokes line for a pair of branches shifts a little bit to the right, then the interference around

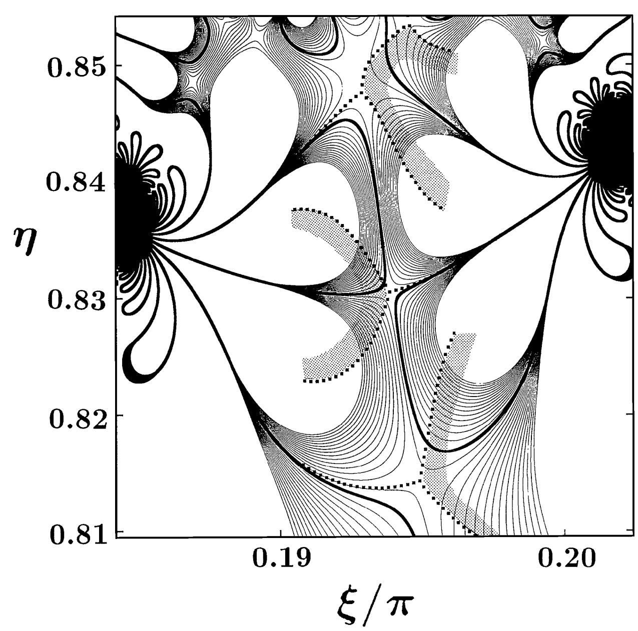

FIG. 2. Stokes lines drawn at typical caustics displayed in Fig. 1. The part of the branch inside the shaded zone together with the higher-order generations following it should be removed as noncontributing.

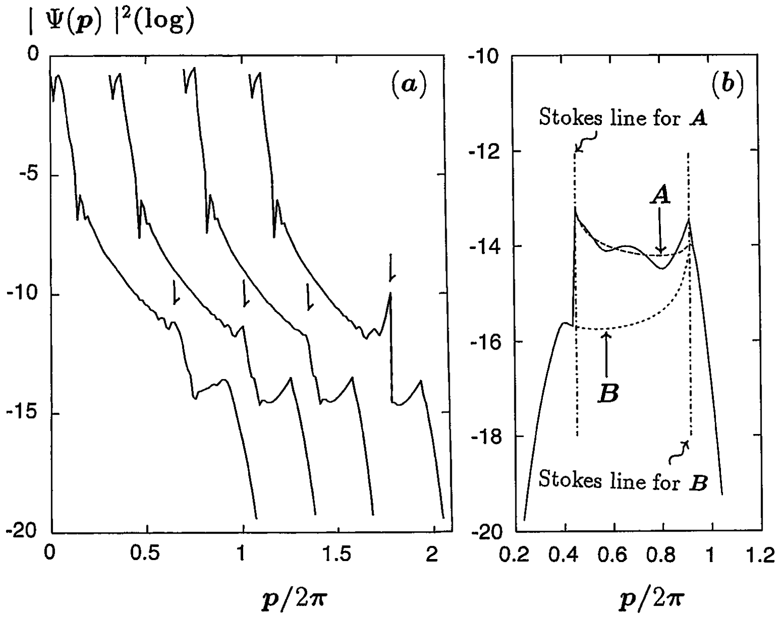

FIG. 3. (a) Tunneling tail of quantum and semiclassical wave functions on a logarithmic scale. The leftmost represents the quantum wave function, and the second is the semiclassical one where the PED is correctly operated. The interference on the shoulder, indicated by arrows, is successfully reproduced. The third is also the semiclassical one, but the Stokes line for the branch giving significant contribution around the shoulder is virtually shifted to the left. The interference pattern on the shoulder disappears. The fourth is also the semiclassical one, but the same Stokes line is virtually shifted to the right. An arrow inserted shows the hump due to such an incorrect operation of PED. (b) Two semiclassical contributions giving dominant contribution to the anomalous tunneling tail, together with their superposition. The broken lines show the two individual contributions ***A*** and ***B***. The solid line is their superposition. The dashed lines with dots represent the Stokes lines determined by PED.

the shoulder disappears while the correct quantum wave function certainly shows such an interference. On the other hand, as also shown in Fig. 3(a), if the same Stokes line shifts to the left, a strange hump at a shoulder of plateau is generated and it must be unphysical.

In the deep tunneling regime far from the classically accessible domain, the PED plays a further important role in describing some anomalous features of the chaotic tunneling tail. Let us focus on the deep tunneling regime in Fig. 3(a). In both quantum and semiclassical wave functions we notice that the tunneling amplitude increases as we go deeper into the tunneling region, while simple tunneling phenomena never show such a strange behavior. This increase abruptly stops and then the amplitude begins to decay rapidly.

The mechanism of this anomalous behavior can be understood by extracting two semiclassical branches giving the most dominant contributions around this p_t region. Figure 3(b) shows two individual semiclassical contributions together with their superposition. The dashed lines with dots represent the position of the Stokes lines which are determined by PED. The dominant contribution comes from branch ***A***, which forms a plateau followed by an abrupt decrease. The key to understanding the anomalous behavior is that the Stokes line cuts branch ***B*** after the associated amplitude begins to increase steeply. Such an increase is led to an exponential explosion and seems as if it is unphysical; however, the anomaly cannot be explained without it. Indeed, the steep increase along branch ***B*** actually manifests itself in a remarkable increase of the summed amplitude, which stops at the Stokes line and takes place with a rapid decay coming from branch ***A***. Such behavior explains exactly the anomalous behavior observed in the tunneling tail. Branch ***B*** can be considered the subdominant contributor, and it is really surprising that the feature of subdominance is visible as an anomalous behavior of the tunneling tail.

In summary, we have proposed a practical method to deal with the Stokes phenomenon appearing in chaotic systems, which can successfully be applied to the chaotic tunneling problem. The method is based upon the presence of a conspicuous tree structure hidden in a complicated set of tunneling branches. We believe that such a remarkable feature is generic in complex classical dynamics and can be utilized in order to unravel the entanglement among complex branches. Indeed, we confirmed that our method works quite well for several kinds of quantum map systems including modified kicked rotors, standard kicked rotors, and so on [10]. In spite of its success, theoretical justification for this practical method is lacking, and its success requires further explanation.

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