Phase Measurement by Projection Synthesis

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Experimental determination of the canonical quantum optical phase probability distribution has, until now, required sufficient measurements to determine the complete state of the field. In this Letter we present a more direct means for measuring this distribution which involves synthesizing the projection onto a phase state. Projection synthesis may be applied more generally to measure the probability distribution associated with other observables. [S0031-9007(96)00342-0]

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The probability, or probability density, for a quantum mechanical observable to be found with a particular value is the expectation value of the projector formed from the eigenstate of that observable with the corresponding eigenvalue. Thus, for example, the photon number probability distribution for a pure state $|f\rangle = \sum c_n |n\rangle$ of an electromagnetic field mode can be obtained reasonably directly by measuring a quantity proportional to $\langle f|n\rangle\langle n|f\rangle$, where $|n\rangle\langle n|$ is a photon number state projector. For weak fields in the quantum domain a suitable quantity to measure is the probability of the release of n photoelectrons by an ideal photodetector. On the other hand, measuring the canonical phase probability distribution, as defined below, is more difficult. For a weak field, the only method presently available appears to be the reconstruction of the entire state by means of either optical homodyne tomography [1] or other related methods [2]. The phase probability distribution and indeed any other probability distribution can then be calculated. This has raised the question [3] of whether or not it is possible, even in principle, to measure the phase distribution more directly. A direct measurement of the canonical phase distribution would involve measuring a quantity proportional to $\langle f|\theta\rangle\langle\theta|f\rangle$ where $|\theta\rangle$ is a phase state which is complementary to the photon number states [4]. We show in this Letter how this can be done.

The probability density for a field in state $|f\rangle$ to have a phase θ is [5]

$$\langle f|\theta\rangle\langle\theta|f\rangle = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} c_n \exp(-in\theta)\right|^2. \tag{1}$$

For any physical state $|f\rangle$, the coefficients c_n must eventually decrease indefinitely with increasing n. It follows that the expectation value (1) can always be approximated to any desired degree of accuracy by setting c_n to zero for $n > N$, if N is suitably large [6]. This allows us to replace (1) with the quantity

$$P_N(\theta) = \frac{1}{2\pi}\left|\sum_{n=0}^{N} c_n \exp(-in\theta)\right|^2. \tag{2}$$

Of course, the error involved in this replacement will be zero for states which are finite superpositions of number states with $c_n = 0$ for all n greater than N. The probability density (2) is proportional to the expectation value of the projector $|\theta, N\rangle\langle\theta, N|$, where $|\theta, N\rangle$ is the truncated phase state [7]

$$|\theta, N\rangle \equiv \frac{1}{\sqrt{N+1}}\sum_{n=0}^{\infty} \exp(-in\theta)\,|n\rangle. \tag{3}$$

All phase and phase-dependent measurements require the introduction of a reference system to set the zero of phase. The simplest way in which this can be achieved is by coherently mixing the field a in state $|f\rangle_a$ with a reference field b, prepared in a state $|B\rangle_b$, by means of a beam splitter as shown in Fig. 1. If $|B\rangle_b$ is a large amplitude coherent state then this arrangement allows us to perform a measurement of a chosen field quadrature [8]. Our task here is to find a suitable reference state $|B\rangle_b$ such that photocounting in the two outputs of the beam splitter leads to the required probability distribution (2). Using the beam splitter, we can measure the probability for finding n and n' photons in the output modes a and b, respectively, which, for ideal photodetectors, is given by ${}_a\langle f|\hat{\Pi}|f\rangle_a$, where $\hat{\Pi}$ is the projector ${}_b\langle B|\hat{R}^\dagger|n'\rangle_b|n\rangle_{aa}\langle n|_b\langle n'|\hat{R}|B\rangle_b$. Here $\hat{R}$ is the unitary transformation linking the output

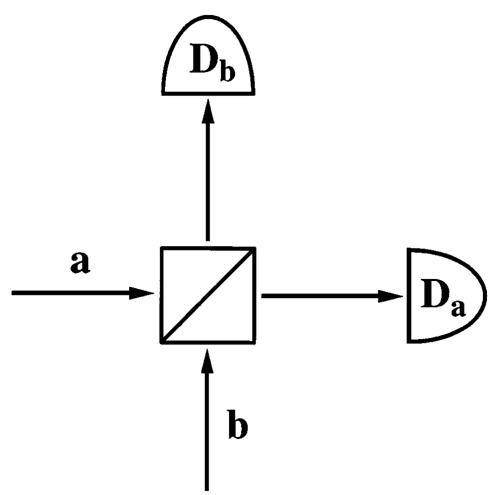

FIG. 1. Schematic representation of a beam splitter with input modes a and b and detectors D_a and D_b.

modes to those for the input [9]. To synthesize a suitable projection which will allow us to find $P_N(\theta)$ by this method, we need to find n, n', $\hat{R}$, and $|B\rangle_b$ such that

$$\hat{\Pi} = K|\theta, N\rangle_{aa}\langle\theta, N|, \tag{4}$$

where K is a positive constant, independent of θ. Remarkably, this problem admits at least one solution.

Consider an ideal 50/50, symmetric beam splitter for which the unitary transformation $\hat{R}$ is [9]

$$\begin{aligned}\hat{R} &= \exp\left[i\frac{\pi}{4}(\hat{b}^\dagger\hat{a} + \hat{a}^\dagger\hat{b})\right] \\ &= \exp(i\hat{b}^\dagger\hat{a})\exp\left[\frac{\ln 2}{2}(\hat{b}^\dagger\hat{b} - \hat{a}^\dagger\hat{a})\right]\exp(i\hat{a}^\dagger\hat{b}).\end{aligned} \tag{5}$$

With this choice, Eq. (4) can be satisfied if we set $n = N$ and $n' = 0$, corresponding to N photocounts in detector D_a and no counts in D_b. The required form of the reference state $|B\rangle_b$ is

$$|B\rangle_b = C\sum_{k=0}^{N}\binom{N}{k}^{-1/2}\exp\left[ik\left(\theta - \frac{\pi}{2}\right)\right]|k\rangle_b, \tag{6}$$

where $\binom{N}{k}$ is the usual binomial coefficient and C is the normalization constant with modulus independent of θ. It is natural to refer to these states as reciprocal-binomial states by analogy with the biminimal states of Stoler, Saleh, and Teich [10]. With this reference state, we find

$$_a\langle N|_b\langle 0|\hat{R}|B\rangle_b = C\exp(iN\theta)2^{-N/2}\sum_{n=0}^{N}{}_a\langle N|\exp(-in\theta), \tag{7}$$

which satisfies Eq. (4) with $K = |C|^2 2^{-N}(N + 1)$. It follows that the probability of registering N counts in D_a and no counts in D_b is proportional to the expectation value of the projector $|\theta, N\rangle\langle\theta, N|$ and hence to the required probability density given in Eq. (2). For a sufficiently large value of N, this yields the phase probability density (1). The full distribution can be obtained by repeating the measurement with reference field states containing different values of θ in Eq. (6). These fields can be prepared by first generating a field in one particular state $|B\rangle_b$ and then changing the value of θ by means of a phase shifter. It is easy to verify that the action of the phase-shift operator $\exp(i\hat{b}^\dagger\hat{b}\Delta\theta)$ on $|B\rangle_b$ is equivalent to adding $\Delta\theta$ to θ.

For simplicity, our analysis has been for a pure state in mode a. It is not difficult to extend this to include any mixed state with density operator $\hat{\rho}_a$. In this case, the probability of registering N counts in D_a and no counts in D_b with reference state $|B\rangle_b$ in Eq. (6) becomes $\mathrm{Tr}(\hat{\rho}_a\hat{\Pi})$, that is, $_a\langle\theta, N|K\hat{\rho}_a|\theta, N\rangle_a$, which is proportional to the probability density of finding the mixed state with phase θ.

In order to perform an experiment it is necessary to select a suitable value for N and then to prepare the required reciprocal-binomial states. The choice of N is determined by the particular state being measured and by the accuracy desired for the measured probability distribution [6]. A practical procedure might be as follows. A preliminary choice of N is made, based either on the mean intensity or on simple knowledge of the source, and the experiment is then performed as described above. The choice of N will lead to an accurate determination of the phase probability distribution if the number of occasions that N counts are registered in D_a and no counts in D_b greatly exceeds those occasions on which the total number of counts registered in the two detectors exceeds $2N$. If this is not the case then N will have to be increased. This procedure ensures that the probability that the field in mode a has more than N photons is sufficiently small for $P_N(\theta)$ to provide an accurate approximation to the phase probability distribution. The probability for registering N counts in D_a with no counts in D_b is then determined as the ratio of the number of these events to the total number of runs. It should be noted that the detection of N counts in D_b and no counts in D_a also provides useful information corresponding to finding the expectation value of the projector $|(\theta - \pi), N\rangle\langle(\theta - \pi), N|$ and hence determining the phase probability density $P_N(\theta - \pi)$. It is only necessary, therefore, to change the phase associated with the reciprocal-binomial states through values in a range of π in order to provide all the information required to reproduce the phase probability distribution. The resulting distribution can be normalized over a 2π range. In our discussion we have, for simplicity, assumed ideal photodetectors. This assumption is not strictly necessary since the ideal detector statistics can be recovered from those measured with sufficiently good detectors [11]. We should note that the experimental procedure described here differs from that used by Noh, Fougères, and Mandel [12]. Their experiments do not measure the canonical phase, that is the complement of the photon number, but rather an operationally defined phase.

Clearly, the most difficult part of the measurement procedure is the preparation of the reciprocal-binomial states. In light of recent work [13], however, it is clear that the problem of generating specific states such as these can and will be solved. Moreover, the realization of measurements based on projector synthesis provides an important motivation for the production of such specially constructed nonclassical states.

Our work answers, in the affirmative, the important question as to whether it is possible in principle to measure the phase distribution without having to obtain sufficient information to reconstruct the complete state. In practice, it would probably be more convenient to measure the photon number probability distribution and then to make a choice of N before applying the projection synthesis technique.

An important application of measuring the phase distribution for a field in a pure state is that, when it is combined with the photon number probability distribution, it provides all the information required to reconstruct the state

[14]. This would provide a possible alternative to existing techniques [1,15] and proposals [2] for experimental state determination. Finally, state projection synthesis provides the means to perform more general measurements than just that of phase. The ability to prepare any chosen reference state for mode b would, in principle, allow the experimental determination of the expectation value of any chosen projector formed from the first $N + 1$ number states.

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