# Two-Loop QCD Corrections to $b \to c$ Transitions at Zero Recoil

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Complete two-loop QCD corrections to $b \to c$ transitions are presented in the limit of zero recoil. Vector and axial-vector coefficients $\eta_{A,V}$ are calculated analytically in the limit of equal beauty and charm masses, and a series appoximation is obtained for the general mass case. $\eta_A$ is crucial for the determination of the absolute value of the Cabibbo-Kobayashi-Maskawa matrix element $V_{cb}$. The two-loop effects enhance the one-loop corrections by 22%, removing a major theoretical uncertainty in the value of $|V_{cb}|$. Including two-loop QCD effects and previously neglected electroweak corrections we find $|V_{cb}| = 0.0383 \pm 0.0021(\text{stat}) \pm 0.0025(\text{syst}) \pm 0.0011(\text{theory})$. [S0031-9007(96)00333-X]

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Elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix are fundamental input parameters of the standard model. Their precise measurements have been the subject of vast experimental efforts and will remain prominent issues in forthcoming projects, most notably the $B$ factories. The values of CKM matrix elements determine the sides of the unitarity triangle, and their precise knowledge is essential for the understanding of the origin of $CP$ violation, a major puzzle of the standard model.

One of the directly measurable CKM parameters is the absolute value of $V_{cb}$. The experimental value can be extracted from decays of $B$ mesons produced on the $\Upsilon(4S)$ resonance (ARGUS [1] and CLEO Collaborations [2]) or on the $Z$ resonance (ALEPH [3] and DELPHI Collaborations [4]).

$|V_{cb}|$ can be obtained either from the total width of semileptonic $B$ decays or from the zero-recoil extrapolation of the exclusive decay spectrum of $B \to D^* l \bar{\nu}$, where $l$ is an electron or muon (see [5] for a recent review). The merits of both methods and theoretical uncertainties have been discussed in Ref. [6]. The inclusive approach has the advantage of larger experimental statistics; the inherent theoretical error is mainly due to inaccurate knowledge of the quark masses which enter the decay width formula. This theoretical uncertainty already dominates the experimental error, and it is not obvious that it can be significantly improved (see, however, a discussion in Ref. [7] and references therein; also, determination of $|V_{cb}|$ from the lepton spectrum in inclusive $B$ decays has been discussed in Ref. [8]).

The exclusive method, on the other hand, benefits from recent advances [6,9] in the heavy quark effective theory (HQET) [10–13]. It has been used to obtain the latest experimental result [4]

$$|V_{cb}| = 0.0385 \pm 0.0021(\text{stat}) \pm 0.0025(\text{syst}) \pm 0.0017(\text{theory}). \tag{1}$$

The exclusive method can be summarized as follows: The recoil spectrum of the $B$ meson decay is

$$\frac{d\Gamma(B \to D^* l \bar{\nu})}{dw} = f(m_B, m_{D^*}, w)|V_{cb}|^2 \mathcal{F}^2(w) \times \left(1 + \frac{\alpha}{\pi} \ln \frac{M_Z}{m_B}\right), \tag{2}$$

where $w$ is the product of the four-velocities of the $B$ and $D^*$ mesons, and $f$ is a known (see, e.g., [9]) function which depends on masses of observable particles (rather than on quark masses). HQET offers a model-independent value of the hadronic matrix element for the decay $B \to D^* l \bar{\nu}$ at zero recoil, $\mathcal{F}(1)$, up to perturbative corrections, to be subsequently discussed. This point is not directly accessible in the experiment due to the vanishing phase space. Fortunately, $|V_{cb}|^2 \mathcal{F}^2(1)$ can be deduced by extrapolating the measured values at nonzero recoil, and, given the theoretical prediction for $\mathcal{F}(1)$, the value of $|V_{cb}|$ can be obtained. The last factor in Eq. (2) approximates the electroweak corrections [14]. In addition, there are long distance QED corrections which differentiate between decays of neutral and charged $B$ mesons. Their difference is given at the rate level by an approximate factor $(1 + \pi\alpha)$ (see [15] and references therein). It represents an enhancement of the $B^0$ decay rate due to the final state interaction between the lepton and the charged $D^*$ meson. To my knowledge, the absolute corrections have not been reliably evaluated and are not being included in the present paper.

The Lorentz structure of the $b \to c$ decay vertex is $\Gamma_\mu = \gamma_\mu(1 - \gamma_5)$. The vector and axial-vector parts are modified in different ways by the QCD corrections; at zero recoil they are parametrized by two functions, $\eta_{V,A}$

$$\gamma_\mu \to \eta_V \gamma_\mu, \qquad \gamma_\mu \gamma_5 \to \eta_A \gamma_\mu \gamma_5 . \tag{3}$$

For the decay $B \to D^* l \bar{\nu}$ only the axial part is relevant. $\eta_V$ is needed, e.g., for the decay $B \to D l \bar{\nu}$. Both functions $\eta_{A,V}$ can be expanded in power series in the strong coupling constant,

$$\eta_{A,V} = 1 + \frac{\alpha_s}{\pi} C_F \eta_{A,V}^{(1)} + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \eta_{A,V}^{(2)} + \mathcal{O}(\alpha_s^3). \tag{4}$$

The one-loop QCD corrections are formally identical to QED effects calculated in the context of muon decay [19].

 

For the heavy quark decays they give [13,20]

$$\eta_A^{(1)} = -\frac{3}{4}\frac{2-\delta}{\delta}\ln(1-\delta) - 2\,,$$
$$\eta_V^{(1)} = -\frac{3}{4}\frac{2-\delta}{\delta}\ln(1-\delta) - \frac{3}{2}\,, \qquad (5)$$

with $\delta = 1 - m_c/m_b$.

The prediction of HQET for $B \to D^*$ transition is free from $1/m_{b,c}$ corrections [16] by virtue of Luke's theorem [17]. The form factor $\mathcal{F}(1)$ can be written as

$$\mathcal{F}(1) = \eta_A(1 + \delta_{1/m^2}). \qquad (6)$$

The mass corrections $\delta_{1/m^2}$ of order $1/m_Q^2$ have been examined [6,18]. They are estimated [18] to decrease the form factor by $(5.5 \pm 2.5)\%$, and their error is responsible for approximately half of the theoretical uncertainty in the value of $|V_{cb}|$ quoted in Eq. (1).

The large remaining theoretical uncertainty is due to the unknown two-loop perturbative QCD corrections. The latter have been the subject of vigorous controversy over the last few years. In the absence of an exact calculation, a renormalization group analysis has been performed [21], but its validity has been questioned in view of the small size of the logarithm of the mass ratio $m_b/m_c$ [22,23]. The need for a full two-loop calculation of $\eta_A^{(2)}$ has been emphasized by many authors [21–24]. The purpose of this paper is to provide this correction.

A calculation of QCD (or even QED) two-loop corrections to a fermion decay is in general very difficult. A full calculation has never been done, neither for the muon nor for a quark. However, it is at present possible to perform such an analysis at least at the zero recoil point. The advantage of this particular kinematical point is twofold: the four-momenta of the decaying and final quarks are parallel; and, because of the phase-space suppression, there is no real radiation.

These two features of the zero-recoil configuration allow an exact analytic solution in the case of equal masses $m_b$ and $m_c$; in the general mass case one can construct an approximate solution in the form of a power series in the relative mass difference $\delta$. In addition, the solution has a very useful symmetry with respect to the exchange $m_b \to m_c$. This nontrivial symmetry, valid only at zero recoil, helps to extract maximum information from the approximating series by accelerating its convergence.

The two-loop QCD diagrams relevant to this calculation are shown in Fig. 1. It is convenient to divide up the functions $\eta_{A,V}^{(2)}$ into parts proportional to various SU(3) factors (an overall factor $C_F$ has been factored out),

$$\eta_{A,V}^{(2)} = C_F\eta_{A,V}^F + (C_A - 2C_F)\eta_{A,V}^{AF} + T_R N_L \eta_{A,V}^L + T_R\eta_{A,V}^H\,. \qquad (7)$$

For a general SU($N$) group $C_A = N$, $C_F = (N^2-1)/2N$, $T_R = 1/2$. $N_L$ denotes the number of the light quark flavors whose masses can be neglected. The last term contains contributions of the massive quark loops, with

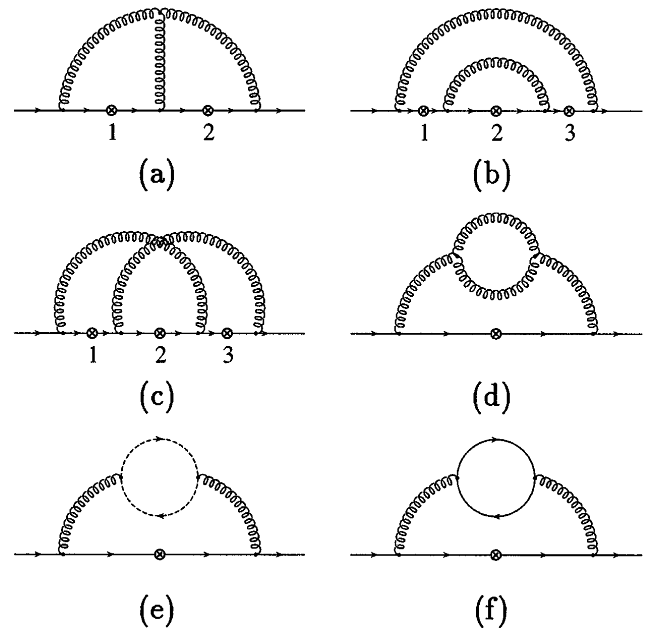

FIG. 1. Two-loop QCD corrections to the $b \to c$ transitions at zero recoil. Symbols ⊗ mark places where the virtual $W$ boson can possibly couple to the quark line. One-particle reducible diagrams are not displayed; they correspond to the renormalization of the quark wave function.

$b$ and $c$ quarks. We neglect the top quark; its impact is suppressed by a factor $\sim m_b^2/m_t^2$.

Among the eight coefficient functions in Eq. (7) $\eta_{A,V}^L$ are already known [25]. They correspond to our diagram (f) in Fig. 1 with a massless fermion in the loop. In the $\overline{\text{MS}}$ scheme (with $\mu = \sqrt{m_b m_c}$), adopted also in the present work, they read

$$\eta_A^L = \frac{5}{24}\left[\frac{2-\delta}{\delta}\ln(1-\delta) + \frac{44}{15}\right],$$
$$\eta_V^L = \frac{1}{24}\left[\frac{2-\delta}{\delta}\ln(1-\delta) + 2\right]. \qquad (8)$$

The remaining six functions can be calculated exactly in the case of equal masses $m_b$ and $m_c$. In this limit the momenta of the leptons in the final state vanish and the vertex function becomes a two-point function with a zero momentum insertion. Such propagatorlike on-shell functions are known; a systematic method of their evaluation has been worked out in Refs. [26,27]; the underlying idea is the integration by parts method [28]. This method has greatly simplified the two-loop QED calculation of $g-2$ [29,30]. In Ref. [29] the computation of two-loop functions with a low number of zero momentum insertions has been automated. For the purpose of the present calculation a new implementation of the recurrence algorithm [31] was necessary; this is because of the necessity of computing two-loop functions with a large number of zero momentum insertions.

In order to go beyond the $m_b = m_c$ limit we use the variable $\delta$ as an expansion parameter. In the real world $m_b$ and $m_c$ are far from being equal; for the purpose of this work we take $m_b = 4.8$ GeV and $m_c = 1.44$ GeV which yields $\delta = 0.7$. Coefficients of the expansion of $\eta_{A,V}$ in $\delta$ are two-point on-shell functions which can be computed using recurrence relations. In order to ensure good numerical accuracy we have computed ten terms in the $\delta$ expansion for all diagrams in Fig. 1. The analytic computation of the resulting integrals was feasible thanks only to the latest achievements in symbolic manipulation programs [32].

The results we obtained are symmetric with respect to the exchange $m_b \leftrightarrow m_c$, or $\delta \to -\delta/(1-\delta)$. The resulting fact that terms with odd powers of $\delta$ can be obtained from the earlier terms provides a strong consistency check of our procedures. On the other hand, it is possible to rewrite the series expansion in a manifestly symmetric form. For this purpose we introduce a variable invariant with respect to $m_b \leftrightarrow m_c$, $\rho \equiv \delta^2/(1-\delta)$. The answer is expected [21,33–35] to contain terms linear and quadratic in $\ln(1-\delta)$. In the variable $\rho$ the radius of convergence of their expansions is $|\rho| = 4$ and corresponds to $|\delta| = 2(\sqrt{2}-1)$ which is less than the original $|\delta| = 1$. However, the physical point $\delta = 0.7$ corresponds to $\rho = 1.633\ldots$ which is well inside the convergence circle and, more important, is positive, whereas the cut starts at $\rho = -4$. Therefore, at the point of interest the series is alternating and the accuracy can be estimated reliably. We obtain accuracy better than 1 per mille even without terms with 5th and higher powers of $\rho$.

For the axial-vector function $\eta_A^{(2)}$ we find

$$\begin{aligned}
\eta_A^{AF} =& -\frac{143}{144} - \frac{1}{12}\pi^2 + \frac{1}{6}\pi^2\ln 2 - \frac{1}{4}\zeta(3) + \rho\left(\frac{29}{576} + \frac{55}{1728}\pi^2 - \frac{1}{16}\pi^2\ln 2 + \frac{3}{32}\zeta(3)\right) \\
&+ \rho^2\left(-\frac{2509}{17\,280} + \frac{121}{8640}\pi^2\right) + \rho^3\left(\frac{43}{22\,680} - \frac{67}{967\,680}\pi^2\right) + \rho^4\left(-\frac{17\,933}{50\,803\,200} + \frac{143}{11\,612\,160}\pi^2\right) \\
\eta_A^{F} =& -\frac{373}{144} + \frac{1}{6}\pi^2 + \rho\left(\frac{377}{576} - \frac{1}{24}\pi^2\right) + \rho^2\left(-\frac{29}{648} + \frac{1}{360}\pi^2\right) + \rho^3\left(\frac{227}{37\,800} - \frac{1}{2520}\pi^2\right) \\
&+ \rho^4\left(-\frac{649}{672\,000} + \frac{1}{15\,120}\pi^2\right) \\
\eta_A^{H} =& \frac{115}{18} - \frac{2}{3}\pi^2 + \rho\left(\frac{529}{72} - \frac{107}{144}\pi^2\right) + \rho^2\left(-\frac{337}{144} + \frac{137}{576}\pi^2\right) + \rho^3\left(-\frac{255\,313}{75\,600} + \frac{197}{576}\pi^2\right) \\
&+ \rho^4\left(-\frac{1\,957\,573}{3\,175\,200} + \frac{1}{16}\pi^2\right).
\end{aligned} \tag{9}$$

For the corrections to the vector current we find

$$\begin{aligned}
\eta_V^{AF} =& \rho\left(\frac{377}{576} - \frac{3}{64}\pi^2 - \frac{1}{16}\pi^2\ln 2 + \frac{3}{32}\zeta(3)\right) + \rho^2\left(-\frac{107}{5760} + \frac{1}{288}\pi^2\right) + \rho^3\left(\frac{41}{10\,080} - \frac{31}{46\,080}\pi^2\right) \\
&+ \rho^4\left(-\frac{13\,927}{16\,934\,400} + \frac{169}{1\,290\,240}\pi^2\right) \\
\eta_V^{F} =& \rho\left(\frac{553}{576} - \frac{5}{72}\pi^2\right) + \rho^2\left(-\frac{227}{4320} + \frac{1}{360}\pi^2\right) + \rho^3\left(\frac{251}{33\,600} - \frac{1}{2520}\pi^2\right) + \rho^4\left(-\frac{7537}{6\,048\,000} + \frac{1}{15\,120}\pi^2\right) \\
\eta_V^{H} =& \rho\left(\frac{197}{72} - \frac{13}{48}\pi^2\right) + \rho^2\left(-\frac{701}{720} + \frac{19}{192}\pi^2\right) + \rho^3\left(-\frac{2851}{1008} + \frac{55}{192}\pi^2\right) + \rho^4\left(-\frac{93\,227}{151\,200} + \frac{1}{16}\pi^2\right).
\end{aligned} \tag{10}$$

We note that all QCD contributions to $\eta_V$ vanish at $m_b = m_c$ ($\rho = 0$), in consequence of vector current conservation.

So far we have not discussed the renormalization procedure which led to the results in Eqs. (9) and (10). For the external quark legs we used the two-loop quark wave function renormalization constant computed in [36]. Vanishing of the terms independent of $\rho$ in Eq. (10) serves as an independent check of the complicated expressions given in [36]. The diagrams (b1) and (b3) require mass counterterms. For these we adopted the on-shell condition. Our results are therefore in terms of the pole masses $m_b$ and $m_c$. For the coupling constant renormalization we used the minimal subtraction scheme ($\overline{\mathrm{MS}}$) condition with the renormalization scale at the geometric mean mass $\mu = \sqrt{m_b m_c}$. It must be noted that the symmetry $m_b \to m_c$ is, in general, valid only for the unrenormalized diagrams. If the coupling constant were normalized at a scale $\mu$ which changed under $m_b \leftrightarrow m_c$, the final result (9) and (10) would not be symmetric.

Numerically, the two-loop corrections evaluate to

$$\begin{aligned}
\eta_A^{(2)} =& -0.586(2)C_F - 0.909(2)(C_A - 2C_F) \\
&+ 0.145T_R N_L - 0.155(4)T_R = -0.944(5),
\end{aligned}$$

$$\eta_V^{(2)} = 0.395(2)C_F - 0.168(2)(C_A - 2C_F) - 0.010T_R N_L + 0.107(2)T_R = 0.509(5). \quad (11)$$

It is interesting to compare these results with an estimate based on the subset of corrections of order $\alpha_s^2\beta_0$ with $\beta_0 = 11 - \frac{2}{3}n_f$. With $n_f = 4$ one gets [25]

$$\eta_A^{(2)} = -0.908, \qquad \eta_V^{(2)} = 0.061. \quad (12)$$

In the axial-vector case the agreement with the full two-loop calculation is quite good. The estimate fails badly in the vector case; this is probably because of an accidental numerical cancellation in (12) which makes the estimate of $\eta_V^{(2)}$ very small. The full one- and two-loop corrections tend to give corrections to $\eta_V$ with approximately half the magnitude of those to $\eta_A$.

Adopting $\alpha_s(\sqrt{m_b m_c}) = 0.24$, our full two-loop calculation leads to the total values of $\eta_{A,V}$

$$\eta_A = 1 - 0.033 - 0.007 + \mathcal{O}(\alpha_s^3) = 0.960 \pm 0.007,$$
$$\eta_V = 1 + 0.018 + 0.004 + \mathcal{O}(\alpha_s^3) = 1.022 \pm 0.004. \quad (13)$$

Since the perturbative series in QCD is asymptotic, the uncertainty in the values of $\eta_{A,V}$ has been estimated by the size of the last computed terms. The central value we obtain for $\eta_A$ is consistent with the value given by Neubert [9], $\eta_A = 0.965 \pm 0.020$, which was adopted in the recent experimental studies. Our result reduces the error bar by a factor of 3 and removes a major source of the theoretical uncertainty in $|V_{cb}|$. This error can perhaps be further decreased by choosing a different renormalization scheme, e.g., the $V$ scheme. This possibility will be examined in a future work. However, with our estimate of the perturbative two-loop corrections to $\eta_A$, the uncertainty in the zero-recoil form factor is dominated by the error in the $1/m_Q^2$ corrections; we adopt here the value $\delta_{1/m^2} = -(5.5 \pm 2.5)\%$ [9,18] (the above result is model dependent; for a recent discussion of these corrections see [37]). Putting this result together with our $\eta_A = 0.960 \pm 0.007$ we find for the zero-recoil form factor

$$\mathcal{F}(1) = \eta_A(1 + \delta_{1/m^2}) = 0.907 \pm 0.026. \quad (14)$$

We use the latest experimental data from the recent DELPHI analysis for the $B^0 \to D^{*-}l^+\nu$ decay rate. We include the electroweak correction, as discussed after Eq. (2). It enhances the rate by 1.3%. Altogether, we find

$$|V_{cb}| = 0.0383 \pm 0.0021(\text{stat}) \pm 0.0025(\text{syst}) \pm 0.0011(\text{theory}). \quad (15)$$

We see that with the decreased theoretical uncertainty further improvement in statistical and systematic accuracy can significantly increase the precision of $|V_{cb}|$ and bring us closer to overconstraining the unitarity triangle.

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