Can Classical and Quantum Variables have a Consistent Mutual Interaction?

In a recent Letter [1], Anderson presents what he describes as "a mathematically consistent scheme for coupling 'classical' and quantum variables." (The author puts "classical" in quotes because "the classical variables evolve to become correlated with the state of the quantum variables. Because this correlation may be with different states in a quantum superposition, the classical variables need not have a definite value but may take a distribution of values depending on the quantum state".) It is the purpose of the present Comment to show that coupling a system which is described classically to a system which is described quantum mechanically leads formally to physical inconsistencies, that is, results that are physically impossible.

As an illustration, I choose a particularly simple example: the behavior of two coupled similar linear oscillators, one described classically and the other quantum mechanically. The Hamiltonian of the combined system is given by $H = H_q + H_{\mathrm{cl}} + H'$, where

$$H_q = (1/2)\hbar\omega(q^2 + p^2), \quad H_{\mathrm{cl}} = (1/2)\hbar\omega(Q^2 + P^2),$$
$$\text{and} \quad H' = \hbar\Gamma(qQ + pP).$$

Here, q and p are the dimensionless coordinate and momentum, respectively, of the quantum oscillator, obeying the commutation rule $[q, p] = i$; Q and P are the corresponding coordinates of the classical oscillator obeying the Poisson bracket relationship $\{Q, P\} = 1$. H_q, H_{cl}, and H' are, respectively, the quantum-oscillator Hamiltonian, the classical-oscillator Hamiltonian (where $\hbar$ has only dimensional significance) and the interaction Hamiltonian (chosen to be of the rotating-wave type for simplicity). The following are the Heisenberg equations for the quantum oscillator and Hamilton's equations for the classical oscillator:

$$\dot{q} = \omega p + \Gamma P, \quad \dot{p} = -\omega q - \Gamma Q,$$
$$\dot{Q} = \omega P + \Gamma p, \quad \dot{P} = -\omega Q - \Gamma q.$$

The solution is given by

$$q(t) = q^{(0)}(t)\cos\Gamma t + P^{(0)}(t)\sin\Gamma t,$$
$$p(t) = p^{(0)}(t)\cos\Gamma t - Q^{(0)}(t)\sin\Gamma t,$$
$$Q(t) = Q^{(0)}(t)\cos\Gamma t + p^{(0)}(t)\sin\Gamma t,$$
$$P(t) = P^{(0)}(t)\cos\Gamma t - q^{(0)}(t)\sin\Gamma t,$$

where the superscript (0) indicates the behavior of the variable in the case of the free, or uncoupled, oscillator ($\Gamma = 0$). Thus,

$$q^{(0)}(t) = q(0)\cos\omega t + p(0)\sin\omega t,$$
$$p^{(0)}(t) = p(0)\cos\omega t - q(0)\sin\omega t,$$

and similarly for $Q^{(0)}(t)$ and $P^{(0)}(t)$. The expressions for the energy of each oscillator are

$$H_q = H_q^{(0)}\cos^2\Gamma t + H_{\mathrm{cl}}^{(0)}\sin^2\Gamma t + K^{(0)}\sin 2\Gamma t,$$
$$H_{\mathrm{cl}} = H_{\mathrm{cl}}^{(0)}\cos^2\Gamma t + H_q^{(0)}\sin^2\Gamma t - K^{(0)}\sin 2\Gamma t,$$

where $K^{(0)} = (1/2)\hbar\omega(q^{(0)}P^{(0)} - Q^{(0)}p^{(0)})$. The time derivatives of H_q and H_{cl} are given by

$$\dot{H}_q = \Gamma(-H_q^{(0)}\sin 2\Gamma t + H_{\mathrm{cl}}^{(0)}\sin 2\Gamma t + 2K^{(0)}\cos 2\Gamma t),$$
$$\dot{H}_{\mathrm{cl}} = \Gamma(-H_{\mathrm{cl}}^{(0)}\sin 2\Gamma t + H_q^{(0)}\sin 2\Gamma t - 2K^{(0)}\cos 2\Gamma t).$$

Consider now initial conditions where both the classical and quantum oscillators are in their respective ground states. This means that $Q(0) = P(0) = 0$, and $\langle H_q^{(0)}\rangle = (1/2)\hbar\omega$, so that

$$\langle\dot{H}_q\rangle = -(1/2)\hbar\omega\Gamma\sin 2\Gamma t,$$
$$\dot{H}_{\mathrm{cl}} = (1/2)\hbar\omega\Gamma\sin 2\Gamma t.$$

Although both oscillators are initially in their ground states and the coupling energy is zero, the quantum oscillator begins to *lose* energy while the classical oscillator begins to *gain* energy. The mathematical—or formal—reason for this physical impossibility is that the classical oscillator sees the zero-point energy of the quantum oscillator as available energy that can be tapped, and the quantum oscillator sees the zero energy of the classical oscillator as a lower energy level to which it can descend. In other words, to the classical oscillator, zero-point oscillations appear to be oscillations that can do work. Had both oscillators been treated quantum mechanically, the zero-point energies would have canceled each other.

One can consider the interaction between dissimilar and more complex systems by means of perturbation theory [2], and come to the same conclusion: The mutual interaction of classical and quantum variables leads formally to physically impossible results when zero-point fluctuations (in effect—the uncertainty principle) cannot be ignored.

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