Multifractal Structure of Auroral Electrojet Index Data

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Using a multifractal approach, based on "singularity analysis," we investigate the scaling properties of the auroral electrojet index (AE) time series. The existence of a multifractal structure in the AE time series is the signature of the occurrence of "intermittence," which can be interpreted as an indication of turbulence in magnetospheric dynamics. Furthermore, a simple model, the *P*-model (a two-scale Cantor set), is shown in order to investigate the underlying multiplicative nature of the signal. This set displays many of the multifractal properties of the AE signal. [S0031-9007(96)00242-6]

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The auroral electrojet index (AE), derived from high latitude fluctuations of the magnetic field horizontal component at Earth's surface, is meant to estimate the total maximum amplitude of the ionospheric current system. It was introduced by Sugiura and Davis [1] to monitor the occurrence of auroral phenomena, and more generally magnetospheric substorms.

The description and modeling of the AE index time series and the study of their scaling properties are powerful tools for understanding the nature of solar wind-magnetosphere-ionosphere coupling and magnetospheric substorm dynamics.

To explain the high temporal variability of magnetic substorms, which is evident in the AE time series (see Fig. 1, top panel), many authors [2–7] investigated the possible occurrence or not of low-dimensional chaos in the magnetospheric response to solar wind input; however, this is still an open question.

Nevertheless, Takalo *et al.* [8–10] have clearly shown the existence of scaling properties in AE index data that suggest that the signal is self-affine, with scaling exponent $H \approx 0.5$, up to a time of about $113(\pm 9)$ min. Furthermore, this characteristic time is well in agreement with the spectral break at $\approx 5.6 \times 10^{-5}$ Hz previously observed by Tsurutani *et al.* [11].

However, the irregularity of AE temporal evolution may suggest a more complex nature of the analyzed phenomenon than that characterized by the above mentioned simple fractal model. The AE time series "spotty" behavior, evidenced when the signal increments are plotted (see Fig. 1, bottom panel), can indeed be an indication of "intermittence," and therefore "turbulence." Furthermore, intermittence involves an anomalous scaling with respect to "time dilation."

This Letter proposes a multifractal approach to the AE time series, based on the so-called "singularity analysis," with the aim of revealing the occurrence of intermittence in the dynamics of magnetospheric substorms.

The purpose of multifractal analysis is to reveal the existence of a hierarchy of scaling indices, which is due to the different local scaling properties of the data. In order to do this, first of all a "positive stationary measure" has to be defined on the data set [12,13]. Since the AE time series power spectral density (PSD) is characterized by power laws with spectral exponent $1 < \beta < 3$ (see Fig. 2), the AE signal is nonstationary with stationary increments over a range of scales which is bounded above

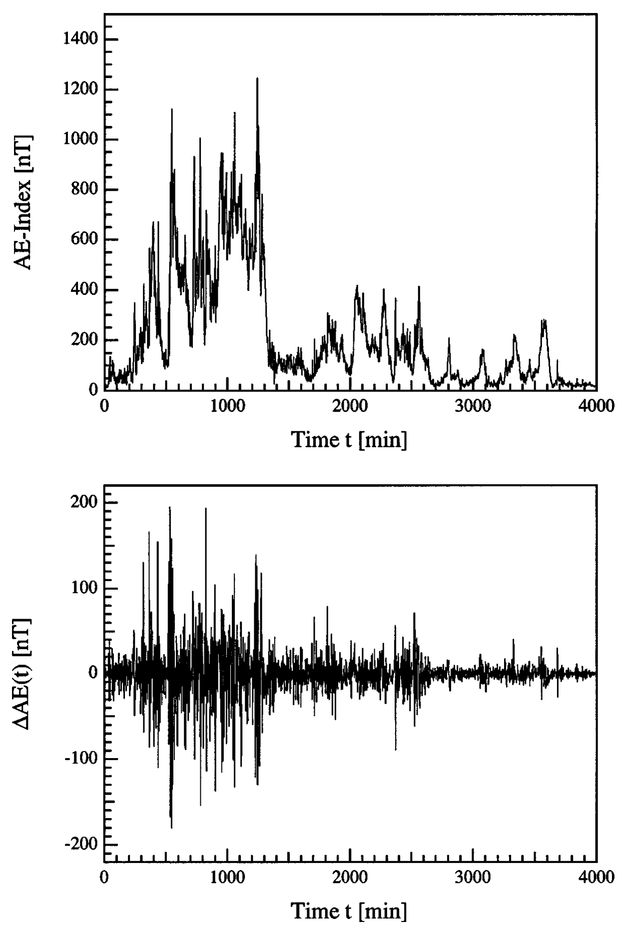

FIG. 1. Sample of the original time series covering a period of three days (top panel) and relative AE increments time series (bottom panel).

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and below [12]. It is important to stress that with the term "stationary" we mean that the data set $\varphi_{\mathrm{AE}}(t)$ is statistically invariant by translation in time t. Therefore, a new scalar stationary, non-negative field $\varepsilon(t)$ has been defined according to Meneveau and Sreenivasan [14], as the squared absolute value of the small scale differences

$$\varepsilon(t) = |\varphi_{\mathrm{AE}}(t_i + \Delta t) - \varphi_{\mathrm{AE}}(t_i)|^2, \tag{1}$$

where $\varphi_{\mathrm{AE}}(t)$ is the original AE-data set, and Δt is the sampling interval. There are several methods to define a stationary non-negative field. However, the peculiar procedure does not affect the results of singularity analysis as Lavallée *et al.* [15] pointed out.

Consequently, a positive measure $d\mu$ can be defined as

$$d\mu(t) = \frac{\varepsilon(t)}{T\langle\varepsilon\rangle}\,dt\,, \tag{2}$$

where T is the total time length. According to Paladin *et al.* [13], a multifractal measure is characterized by the scaling features of its coarse-grained weight:

$$p_i(\tau) = \int_{\Lambda_i} d\mu(t) \approx \sum_{t_i \le t' \le t_i+\tau} \Delta\mu(t')\,, \tag{3}$$

where $\tau = 2^n$ is the size of the segments Λ_i. The presence of multifractality is shown by the anomalous scaling of the "partition function $\Gamma(q,\tau)$" for small τ:

$$\Gamma(q,\tau) = \sum_{\Lambda_i} p_i(\tau)^q \approx \tau^{\gamma(q)}, \tag{4}$$

where $\gamma(q) = (q-1)D_q$ and D_q is a nonconstant function. The exponents D_q, called "generalized dimension" [16], are independent of moment order in the case of homogeneous fractality.

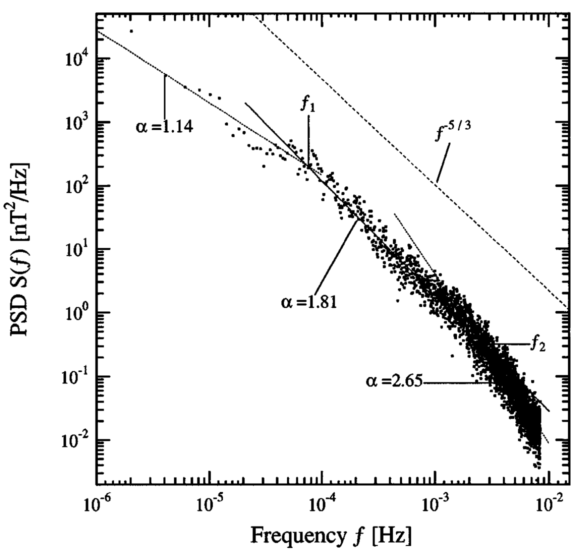

FIG. 2. *Power spectral density* (PSD) or *energy spectrum* relative to the period under analysis. The solid and dotted lines are power-law best fits. The dashed line is relative to $-5/3$ power law predicted for the "inertial range" by Kolmogorov's theory of fully developed turbulence in absence of "intermittence." Two spectral breaks, f_1 and f_2, identify a frequency region where the dependence is nearly similar to Kolmogorov's one.

We developed our analysis applying the multifractal approach to a set of AE-index data, covering the period from 1.1.1975 to 19.2.1975, with 1-min time resolution, for a total amount of 2^{16} points. Data comes from the National Geophysical Data Center, Boulder, CO.

To evaluate the exponents $\gamma(q)$, the partition function $\Gamma(q,\tau)$ vs τ has been fitted with a power law using the Levemberg-Marquardt nonlinear regression algorithm [17].

In Figs. 3 and 4 $\gamma(q)$ and D_q are plotted as a function of the moment order q. It is evident that $\gamma(q)$ and q are not linearly dependent; this is the consequence of an underlying multifractal structure in the AE signal, as is also confirmed by the existence of a hierarchy of dimension D_q.

The existence of a multifractal nature of the AE signal in respect to time dilation is the signature of temporal inhomogeneity, or, in other words, of intermittence [13].

The occurrence and the nature of intermittence in the AE signal have been further analyzed by comparing the D_q curve with those proposed for two typical multiplicative processes, the *P-model* [14] and the *Log-normal model* [18]. These models were first introduced to account for the occurrence of intermittence in fully developed turbulence in ordinary fluid flows.

The solid line in Fig. 4 is the nonlinear best fit of the D_q data according to the *P-model*, which is formally equivalent to a "two-scale Cantour set" with $l_1 = l_2 = 1/2$ and represented by

$$D_q = \log_2[p^q + (1-p)^q]^{1/(1-q)}, \tag{5}$$

where p is a parameter, associated with the fragmentation probability in the cascade process, and q is the moment order.

The dash-dotted line represents D_q behavior according to the *log-normal model*. There is agreement between this model and D_q data only if small values of q are

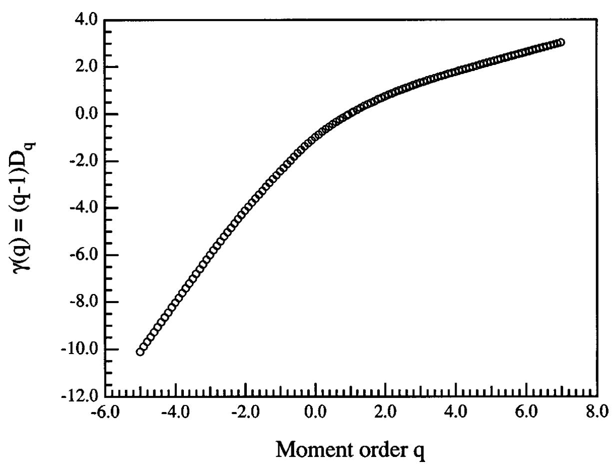

FIG. 3. Scaling exponent $\gamma(q)$ of the partition function $\Gamma(q,\tau)$ as a function of moment order q.

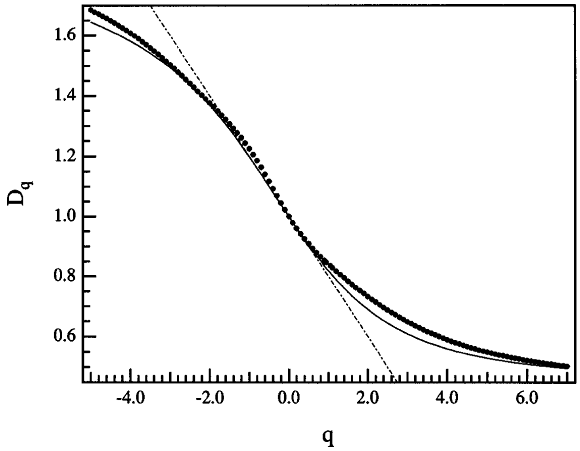

FIG. 4. Plot of *generalized dimensions* D_q. The solid line is the nonlinear best fit of the data by the *P-model* (see text). The dash-dotted line represents the behavior of the *Log-normal model*.

considered. On the contrary, the experimental D_q curve and the *P-model* best fit are fairly well in agreement.

The agreement between D_q data and *P-model* prediction must be interpreted as the evidence for partial mixing during the cascade and for an asymmetric breakdown in the fragmentation process. The parameter *p*, as evaluated from the nonlinear best fit of D_q data using relation (5) gives

$$p = 0.746 \pm 0.002 . \tag{6}$$

This parameter can be used to evaluate the intermittence coefficient μ:

$$\mu = -2 \frac{dD_q}{dq} |_{q=0} = \log_2 [4p(1 - p)]^{-1} = 0.400 \pm 0.002 . \tag{7}$$

In the case of homogeneous turbulence, Kolmogorov's theory (via Taylor's hypothesis) predicts a $f^{-\beta}$ law, with $\beta = 5/3$, in the *inertial range* of energy spectrum. When intermittence is considered the exponent β must be corrected as follows:

$$\beta \Rightarrow \beta + \frac{\mu}{3} = \alpha , \tag{8}$$

where μ is the intermittence coefficient. From this we obtain

$$\alpha = 1.800 \pm 0.001 , \tag{9}$$

which is well in agreement with the PSD power-law exponent when the intermediate range $[f_1, f_2]$,

$$7.3 \times 10^{-5} < f < 2.5 \times 10^{-3} \text{ Hz} , \tag{10}$$

is considered (see Fig. 2).

Another way to characterize the multifractality is given by the so-called *multifractal* or *singularity spectrum* $f(\alpha)$, which can be directly evaluated from the $\gamma(q)$ curve by a Legendre transformation:

$$\alpha = \frac{d\gamma(q)}{dq} \qquad f(\alpha) = q\alpha - \gamma(q) . \tag{11}$$

In Fig. 5 we report the *singularity spectrum* $f(\alpha)$ derived for the AE index data set (pointed curve). The solid line represents a nonlinear best fit if an analytical expression for the *P-model* is used, which can be derived from that of a general two-scale Cantor set with equal scales ($l_1 = l_2 = 1/2$) but unequal weights [19],

$$\alpha = -\frac{\log_2 p + (n/m - 1) \log_2 (1 - p)}{n/m}$$
$$f(\alpha) = -\frac{(n/m - 1) \log_2 (n/m - 1) - (n/m) \log_2 (n/m)}{n/m} \tag{12}$$

eliminating n/m. Once again there is good agreement between theory and data.

The multifractal approach in respect to time dilation has evidenced the existence of different local scaling properties in the AE-time series. This is a consequence of temporal inhomogeneity that is related to the intermittent character of the signal.

The comparison between the multifractal structure of the AE signal and that of two typical multiplicative processes, introduced in order to explain the intermittence in turbulence, has clearly shown that the nature of the signal is analogous to intermittent turbulence. It is important to stress that intermittent turbulence involves a different energy distribution in space and time from the prediction of Kolmogorov's theory for ordinary turbulence. Moreover the multifractal structure of the AE index seems to be analogous to the *P-model* prediction.

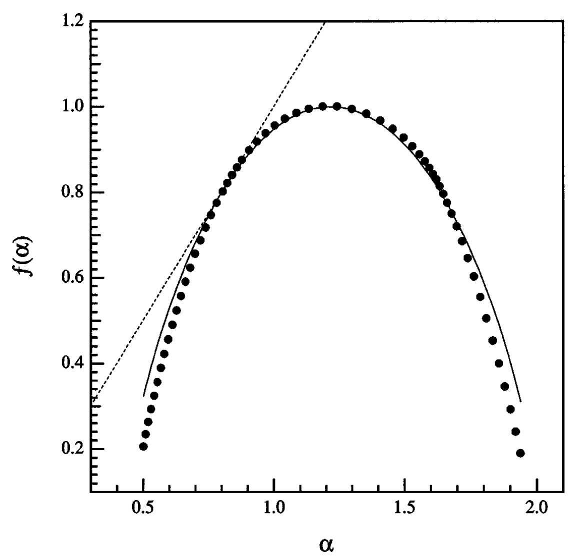

FIG. 5. *Multifractal or singularity spectrum* as derived from Legendre transformation. The solid line is a nonlinear regression best fit of the data making use of an analytical expression for a *two-scale Cantor set* with equal scales but unequal weights (*P-model*). The dashed line is the diagonal which indicates the homogeneous fractal locus.

Furthermore, starting from the intermittence coefficient μ, evaluated on the basis of the *P-model*, it was possible to identify a spectral region in which the PSD is well in agreement with Kolmogorov's spectrum corrected in the case of intermittence.

As a consequence of this analysis, we can conclude that intermittence and turbulence must be considered as a much more relevant phenomena than low-dimensional chaos in the evolution of magnetic substorms. Furthermore, a pure mathematical model, the *P-model*, seems to be able to explain the underlying multifractal structure of the signal, and this is certainly useful information in the elaboration of a magnetospheric model.

Further studies are necessary in order to extend this analysis to other periods and to explain the AE intermittent and turbulent character within a general magnetospheric-ionospheric model.

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