Exponential Growth of the Energy of a Wave in a 1D Vibrating Cavity: Application to the Quantum Vacuum

O. Méplan and C. Gignoux
Institut des Sciences Nucléaires, IN2P3-CNRS/UJF G, 38026 Grenoble, France.
(Received 16 March 1995)

The wave equation in a one–dimensional cavity is equivalent to the motion of *massless particles* in a two-dimensional space-time billiard. This allows us to consider, in a simple way, the case of a cavity with one or two oscillating walls. It is shown that a set of continuous families of frequencies of the oscillating walls leads to an exponential growth of the energy of a wave. For such cases, the wave energy is localized in narrow space regions moving at the wave velocity. As a consequence the electromagnetic vacuum fluctuations inside the cavity increase exponentially and the possibility to observe experimentally photon creation from the vacuum could be reconsidered.

PACS numbers: 42.50.Lc, 03.40.Kf, 03.50.–z, 03.65.Ge

It is well known since Casimir's work that the fluctuations of the electromagnetic field inside a perfectly reflecting cavity generate a measurable attractive force. However, only few results are known for the case when one of the walls, or mirrors, oscillates. The standard method for solving this problem has been developed by Moore [1] and consists in finding a function $R(z)$ satisfying

$$R(t + L(t)) = R(t - L(t)) + 2\,, \tag{1}$$

where $L(t)$ is the distance between the moving and the fixed mirror.

Few analytical solutions are known and it is only recently that Law [2] gave one for an almost sinusoidal movement of the mirror. In this special case, he has shown that the quantum energy density shrank into two packets and that energy increased quadratically with the time. At the same time, Dittrich *et al.* [3] have found the generic time behavior for the energy.

A completely different approach which has a transparent geometric interpretation is presented here. This method permits one to predict the time energy behavior for any kind of motion for the *two* mirrors. Although the problem under consideration could be applied to any physical situation described by the wave equation (elastoacoustic, string, etc.), we will derive our results with the example of the electromagnetic field. Let us consider the electromagnetic vector potential $A(x,t)$ which obeys

$$(-\partial_t^2 + \partial_x^2)A(x,t) = 0\,. \tag{2}$$

We have set the wave velocity $c = 1$. We introduce the one-dimensional free forward propagator,

$$K_0(x - x_0, \tau) = -\frac{1}{2}\Theta(\tau - |x - x_0|)\,,$$

where $\tau = t - t_0$ and $\Theta(t)$ is the Heaviside function. This propagator satisfies

$$(-\partial_t^2 + \partial_x^2)K_0(x - x_0, \tau) = \delta(x - x_0)\delta(\tau)\,. \tag{3}$$

Generalizing the Korringa-Kohn-Rostoker method [4] for a space-time boundary, Eqs. (2) and (3) permit us to write $A(x_0,t_0)$ as the flux of the vector $\mathbf{j} = (A \overleftrightarrow{\partial}_x K_0, -\overleftrightarrow{A}\,\partial_t K_0)$ in the two-dimensional space-time through any surface surrounding (x_0,t_0) $(f \overleftrightarrow{\partial} g = f\partial g - g\partial f)$.

For clarity and without loss of generality, let us consider the case of only one moving mirror with the simple time-dependent boundary conditions $A(0,t) \equiv A(L(t),t) \equiv 0$.

With these Dirichlet boundary conditions, we obtain

$$A(x_0,t_0) = A^0(x_0,t_0) + \frac{1}{2}\left\{\int_0^{t_l} \mu_0(t)\,dt + \int_0^{t_r} [1 + \dot{L}(t)]\mu_L(t)\,dt\right\}, \tag{4}$$

where $A^0(x_0,t_0)$ is the (free) d'Alembert's solution, and μ_L and μ_0 are functions to be defined later.

The times $t_{r/l}$ are the intersections [when they exist, otherwise $t_{r/l} = 0$, see Fig. 1(a)] of the downward light cone, with apex at (x_0,t_0), with the right $[x = L(t)]$ and left $(x = 0)$ boundaries. It is clear from (4) that knowledge of $\mu(t)$ is sufficient to construct the solution $A(x_0,t_0)$. These functions, $\mu_0(t) = -\partial_x A(x,t)|_{x=0}$ and

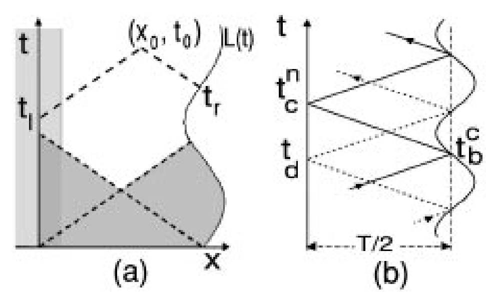

FIG. 1. (a) The wave $A(x_0,t_0)$ is constructed from its x derivatives on mirrors between $[0,t_l]$ and $[0,t_r]$ and from d'Alembert's solution, which is relevant only in the grey dark part. The dashed lines represent geodesics. The pale grey rectangle corresponds to one family of positions of the fixed mirror leading to exponential growth of energy. (b) The two periodic trajectories corresponding to the fixed points of (7) and (8) for $p = 1$; the other trajectories will converge on the solid line.

0031-9007/96/76(3)/408(3)$06.00

$\mu_L(t) = [1 - \dot{L}(t)]\, \partial_x A(x,t)|_{x=L(t)}$, are defined on the surface of the fixed and moving mirror, respectively; in the electromagnetic case, the two integrals in (4) represent the time integrals of the magnetic field, i.e., the surface current, in the proper frame of each mirror ($\dot{L}$ is the mirror velocity, $|\dot{L}| < 1$).

Differentiating (4) on each boundary, we obtain

$$\begin{aligned} \mu_0(t_0) &= -(\partial_x + \partial_t)A(x,t)|_{t=0,x=t_0}\,, \\ \mu_L(t_0) &= (\partial_x - \partial_t)A(x,t)|_{t=0,x=L(t_0)-t_0}\,, \end{aligned} \tag{5}$$

unless $t_0 - t_r = L(t_r)$ and $t_0 - t_l = L(t_0)$ have a solution, in which case (5) is replaced by

$$\mu_0(t_0) = -\mu_L(t_r)\,, \tag{6a}$$

$$\mu_L(t_0) = -[1 - \dot{L}(t_0)]/[1 + \dot{L}(t_0)]\mu_0(t_1)\,. \tag{6b}$$

When Eq. (5) holds, μ_L and μ_0 represent, respectively, the influence on each mirror of the right and left traveling initial wave packets (which are functions of $x \mp t$, respectively).

Equations (6) and (5) have a classical interpretation. The wave packet evolution can be viewed as the evolution of two *beams of massless, noninteracting particles* leaving at $t = 0$, one to the right, one to the left with the wave velocity. We associate with these particles a scalar value μ, *the momentum* or *the energy*. They are "launched" from the segment $[0, L(0)]$ with an initial momentum μ that they will keep until the first contact on the mirrors: These μ are given by (5). Moreover, each ray of the beam carries a number of particles equal to this initial momentum. The particles rebound elastically on each mirror. Equation (6a) corresponds to momentum conservation on the fixed boundary and the multiplicative factor in (6b) is the result of the Doppler effect. We are now able to construct the functions μ [knowledge of $A(x,t)$ is useless because of the gauge invariance]. We can rewrite (6) as

$$\mu_{n+1} = \kappa(t_b)\mu_n\,, \tag{7}$$

where $\kappa(t_b) = [1 - \dot{L}(t_b)]/[1 + \dot{L}(t_b)]$ and μ_n is the value of μ_0 on the fixed mirror at the nth collision at time t_n; $t_{n+1}(t_n)$ is obtained by solving the equations

$$\begin{aligned} t_{n+1} &= t_b + L(t_b)\,, \\ t_n &= t_b - L(t_b)\,, \end{aligned} \tag{8}$$

where t_b is the time at which the particles of the same ray impinge on the moving mirror. Equations (7) and (8) define an area-preserving mapping [5].

As an illustration, let us now consider a simple periodic motion of the mirror (e.g., sinusoidal). We take particular interest in two hyperbolic fixed points of the mapping (7) and (8) given by ($\mu_0 = 0, t_d \bmod pT$) and ($\mu_0 = 0, t_c \bmod pT$). The first corresponds to a *contraction* in μ and *dilatation* in t and conversely for the second. The corresponding periodic "trajectories" of period pT (T is the period of the mirror oscillations and p a positive integer) are those of a fixed cavity [see Fig. 1(b)]. Clearly, these two points exist only if $L_{\min} < p(T/2) < L_{\max}$. Under this condition, the system exhibits a very singular behavior: Starting at $t = 0$ with $\mu_0 \neq 0$, the energy (μ) will inexorably increase because it will be attracted by the hyperbolic asymptote, leading to an exponential dilatation of μ (while its time support decreases in the same way). Indeed, let t_c^n be the nth bounce on the fixed mirror of the pT-periodic trajectory and denote by τ the dispersion of the wave packet around t_c^{n+1}; from (7), on one hand, and from the area-preserving map, on the other hand, we can state that $\mu_{n+1}(t_c^{n+1} + \tau) = \kappa(t_b^c)\mu_n[t_c^n + \kappa(t_b^c)\tau]$ in the asymptotic regime where $t_b^c = t_c + pT/2$.

The wave energy density $\epsilon(x_0, t_0)$ is obtained from (4) and (6) in terms of the function μ_0 only as

$$\begin{aligned} \epsilon(x_0, t_0) &= \frac{1}{2}\,(|\partial_x A|^2 + |\partial_t A|^2) \\ &= \frac{1}{4}\,[\mu_0^2(t_0 - x_0) + \mu_0^2(t_0 + x_0)]. \end{aligned}$$

In this resonant case, for increasing time, the localization of the beam on the periodic trajectory [corresponding to the fixed point ($\mu_0 = 0, t_c$)] induces a localization of energy on this trajectory; hence the contribution to $\epsilon(x_0, t_0)$ is significant only if $t_0 \pm x_0(\mathrm{mod}T)$ is in the neighborhood of t_c.

The total energy $E(t_0)$ is transformed to an integral of μ_0^2 over time from $t_0 - L(t_0)$ to $t_0 + L(t_0)$. Since $\mu_0^2\, dt$ appears in the energy, the ratio $E(t_c^{n+1})/E(t_c^n)$ tends to $\kappa(t_b^c)$; this leads to *an exponential increase of energy* with the time (the same result was found in a different way in [3]). More precisely, the energy increases in geometric progression step by step, each time the (localized) wave packet hits the moving mirror (see Fig. 2). It is worthwhile to note that in the massless-particle interpretation the energy of the wave is the sum of the energy of each particle which increases by a factor κ at each collision on the moving mirror. This interpretation is also convenient for finding the force on the mirrors. This enhances the analogy between the two views.

Let us consider the quantum case. Following Moore [1], all physical quantities are deduced from the derivative of a function $R(z)$ defined by (1), i.e., $[1 + \dot{L}(t)]\dot{R}(t + L(t)) = [1 - \dot{L}(t)]\dot{R}(t - L(t))$: This is exactly (6) if we identify $\dot{R}(t)$ to $\mu_0(t)$. Fulling and Davies [6] have introduced a general function f defined by

$$24\pi f = \frac{\dddot{R}}{\dot{R}} - \frac{3}{2}\left(\frac{\ddot{R}}{\dot{R}}\right)^2 + \frac{\pi^2}{2}\dot{R}^2$$

in order to write the regularized density energy of vacuum as

$$\langle T_{00}\rangle_{\mathrm{reg}} = -[f(t_0 + x_0) + f(t_0 - x_0)]\,,$$

which is similar to the $\epsilon(x_0, t_0)$ we found in the classical case. Thus, when $t \simeq t_c^n$, $\dot{R}$, $\ddot{R}$, and $\dddot{R}$ are multiplied by κ, κ^2, and κ^3, respectively, and thus energy density

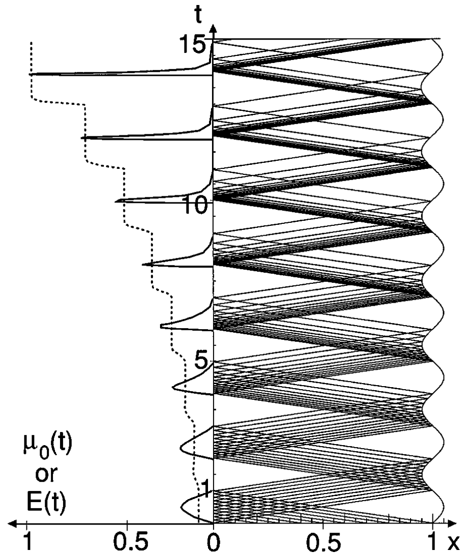

FIG. 2. Trajectories of massless particles corresponding to the mapping (6) for a resonating case; for clarity, we represent (right-hand part) only particles corresponding to the left traveling initial wave packet. Concentration of particles on one periodic trajectory show the localization of energy. The left-hand part shows (solid line) the energy μ_0 associated with each particle. The dashed line represents the energy of the wave (or particles).

increases with a geometric ratio κ^2; owing to the localization of energy, or more precisely, of the vacuum fluctuations (in a wave packet oscillating between mirrors), the energy increases with a ratio κ.

The correspondence between waves and massless particles permits us to generalize immediately to the situation of two vibrating mirrors. The quality of the mirrors can also be taken into account by imposing the mixed Neumann-Dirichlet boundary conditions on the field to obtain an equation similar to (4). The same result could be obtained in a simpler way by considering a loss of the massless particles at each bounce. In summary, the behavior of the energy in a cavity is related to the existence of periodic trajectories in a space-time billiard.

The situation described in [2] deals only with the limiting case $T = L_{\max}$, i.e., $\kappa = 1$ and $p = 2$. The two $2T$-periodic trajectories (the second one is just a translation of T of the first one), hitting the moving mirror at its maximum separation where $\dot{L}(t_b^c) = 0$, attract all the other trajectories. When the localization is almost achieved, one can show that the mapping (7) and (8) leads to a quadratic energy growth.

The resonance condition can be generalized: If $L_{\min} < (p/q)T/2 < L_{\max}$ (q is the number of bounces on the fixed wall during pT) then the mapping (7) and (8) can have an even number of hyperbolic fixed points. Half of these points attracts all the other trajectories and leads to localization; when $q = 1$ these points always exist. For the simplest case of two oscillating mirrors, i.e., a one-dimensional shaken cavity, the condition of resonance is rewritten $L_0 - 2\alpha < (p/q)T/2 < L_0 + 2\alpha$, where L_0 is the length of the cavity and 2α is the displacement.

In fact, for any general periodic motion of the mirrors, it is possible to find a set of continuous frequencies leading to exponential growth and localization of energy both for classical case and for quantum fluctuations of vacuum. The behavior of nonrelativistic massive particles in a one-dimensional billiard (the so-called Fermi accelerator) does not present such resonances, neither in classical nor quantum mechanics [7,8].

This localization of the vacuum fluctuations has been observed by Slusher for the squeezed states of light (see the review article [9]). The exponential growth of the energy gives a new motivation for reconsidering the experimental situation already regarded in [10] for observing the photon creation from vacuum. Indeed, the approach we have presented here may now make experiments more feasible for a number of reasons. First, imperfections of mirrors can be estimated; in addition, we can treat more general motions of the two mirrors. Moreover, and perhaps most importantly, the oscillation frequency can be lower than natural frequency of the static cavity ($q \gg 1$) and the resonance width can be calculated.

We thank Professor R. Arvieu, Professor J. Carbonell, Professor K. Protasov, and Professor W. Roberts for their constant interest and Professor J. Derouard for discussions on the experimental aspect.

[1] G. Moore, J. Math. Phys. **11**, 2679 (1970).
[2] C. Law, Phys. Rev. Lett. **73**, 1931 (1994).
[3] P. D. J. Dittrich, P. Duclos, and P. Šeba, Phys. Rev. E **49**, 3535 (1994).
[4] M. Berry, Ann. Phys. **131**, 163 (1981).
[5] I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge University Press, London, 1982).
[6] S. Fulling and P. Davies, Proc. R. Soc. London A **348**, 393 (1976).
[7] A. J. Lichtenberg and M. A. Lieberman, in *Regular and Stochastic Motion*, Applied Mathematical Sciences Vol. 38 (Springer-Verlag, New York, 1983).
[8] G. Chu and J. V. José, J. Stat. Phys. **68**, 153 (1992).
[9] R. Slusher and B. Yurke, Sci. Am. **258**, No. 5, 50 (1988).
[10] E. Yablonovitch, Phys. Rev. Lett. **62**, 1742 (1989).