Universality of Flux Creep in Superconductors with Arbitrary Shape and Current-Voltage Law

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The nonlinear and nonlocal diffusion equation for the relaxing current density $J(\mathbf{r},t)$ in long superconductors of arbitrary cross section in a constant perpendicular magnetic field B_a is solved exactly by separation of variables in the electric field $E(\mathbf{r},t) = f(\mathbf{r})g(t)$. This solution includes the limiting cases of longitudinal and transverse geometries and applies to the current-voltage laws $E \propto J^n$ ranging from Ohmic ($n = 1$) to Bean-like ($n \to \infty$) behavior. The electric field profile $f(\mathbf{r})$ weakly depends on n and becomes universal for n exceeding ≈ 5. At large times t one finds $E \propto 1/t^{n/(n-1)}$ and $J \propto 1/t^{1/(n-1)}$ for $n > 1$, and $E \propto J \propto \exp(-t/\tau_0)$ for $n = 1$. The contour lines of the creeping $E(\mathbf{r},t)$ coincide with the field lines of $\mathbf{B}(\mathbf{r},t)$ in the remanent state $B_a = 0$. [S0031-9007(96)00286-4]

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Type-II superconductors, in particular the high-T_c superconductors, in general are not ideal conductors of electric current when put into a magnetic field B_a. Nonzero resistivity appears when the Abrikosov vortices induced by B_a depin due to a current density J exceeding a critical value J_c, or by thermal activation occurring also at $J < J_c$. The resulting motion of vortices generates an electric field $E = Bv$, where B is the flux density and v the drift velocity. A superconductor may thus be characterized by its current-voltage law $E(J)$, which in general is nonlinear and strongly depends on B_a and on the temperature T. This $E(J)$ characteristic is most sensitively extracted from magnetization measurements, e.g., by slowly increasing or decreasing $B_a(t)$ and then keeping it at a constant (or zero) value at times $t \geq 0$. During such creep experiments, the magnetic moment caused by persistent currents relaxes with an approximately logarithmic time law when $E(J)$ is strongly nonlinear, and exponentially when $E(J) = \rho J$ is linear (Ohmic).

Recently, Gurevich and co-workers have shown that in both longitudinal [1,2] and transverse [2,3] geometries, corresponding to zero and unity demagnetizing factors, respectively, the relaxing electric field $E(\mathbf{r},t)$ is *universal* if $E(J)$ is sufficiently nonlinear. This means that after some transient time t_1, $E(\mathbf{r},t) = f(\mathbf{r})g(t)$ separates into a universal profile $f(\mathbf{r})$ and a time dependence $g(t) \propto 1/(t_1 + t) \approx 1/t$. The relaxing current density $J(\mathbf{r},t)$ and magnetic moment $\mathbf{m}(t) = \frac{1}{2}\int \mathbf{r} \times \mathbf{J}(\mathbf{r},t)d^3r$ are then obtained by inserting this general electric field into the material law $J = J(E)$. This separation was found to be exact if one has $\partial E/\partial J = E/J_1$ with constant J_1, corresponding to an exponential law $E(J) = E_c \exp[(J - J_c)/J_1]$, and if E depends only on one spatial coordinate. This applies to slabs [1,2] or cylinders [4] in parallel field, and to thin strips or circular disks in perpendicular field [2,3]. The same universality of creep, and the existence in transverse geometry of "neutral lines" along which the flux-density $B(\mathbf{r},t)$ stays constant during creep, was recently demonstrated by magneto-optic observation and by computation to occur also in thin specimens of square shape [5].

In the present Letter I show that the exact solution of the creep problem by separation of variables of $E(\mathbf{r},t)$ is possible for a much more general geometry including specimens of finite thickness and arbitrary cross section in a perpendicular field, and for all current-voltage laws $E(J) \propto J^n$, or, more precisely,

$$E(J) = E_c|J/J_c|^n \,\mathrm{sgn} J\,, \qquad (1)$$

yielding $J(E) = J_c|E/E_c|^{1/n} \,\mathrm{sgn} E$. This power law is observed in numerous experiments and was considered elsewhere in connection with creep [2,6] and flux penetration and ac susceptibility [4,7–10]. It corresponds to a logarithmic current dependence of the activation energy $U(J) = U_c \ln(J_c/J)$, which inserted into an Arrhenius law yields $E(J) = E_c \exp(-U/kT) = E_c(J/J_c)^n$ with $n = U_c/kT \gg 1$; it contains only two parameters E_c/J_c^n and n; and it interpolates from Ohmic behavior ($n = 1$) over typical creep behavior ($n = 10,\ldots,20$) to "hard" superconductors with Bean-like behavior ($n \to \infty$). A power law avoids the unphysical behavior at $J = 0$ exhibited by the exponential law originally used in Refs. [1–3] to achieve separation of variables in the creep problem.

Consider the rather general geometry of a long (along z) superconductor of arbitrary cross section S in a perpendicular field B_a directed along y, e.g., a rectangular bar filling the volume $|x| \leq a$, $|y| \leq b$, $|z| < \infty$, cf. Fig. 1. The current density $J(x,y)$, electric field $E(x,y)$, and vector potential $A(x,y)$ then point along z and the magnetic induction $\mathbf{B}(x,y) = \nabla \times \mathbf{A}$ is in the x-y plane. Throughout this Letter I shall assume $\mathbf{B} = \mu_0\mathbf{H}$, i.e., disregard the finite value of the lower critical field B_{c1}. This approximation is excellent when everywhere inside the superconductor one has $B > 2B_{c1}$. From $\mathbf{J} = \nabla \times \mathbf{H} = \mu_0^{-1}\nabla \times (\nabla \times \mathbf{A})$ one then gets for our geometry $J = -\mu_0^{-1}\nabla^2 A$, which has the solution

$$A(\mathbf{r}) = -\mu_0 \int_S d^2r'\, Q(\mathbf{r},\mathbf{r}')J(\mathbf{r}') - xB_a\,,$$
$$Q(\mathbf{r},\mathbf{r}') = (1/2\pi)\ln|\mathbf{r} - \mathbf{r}'|\,. \qquad (2)$$

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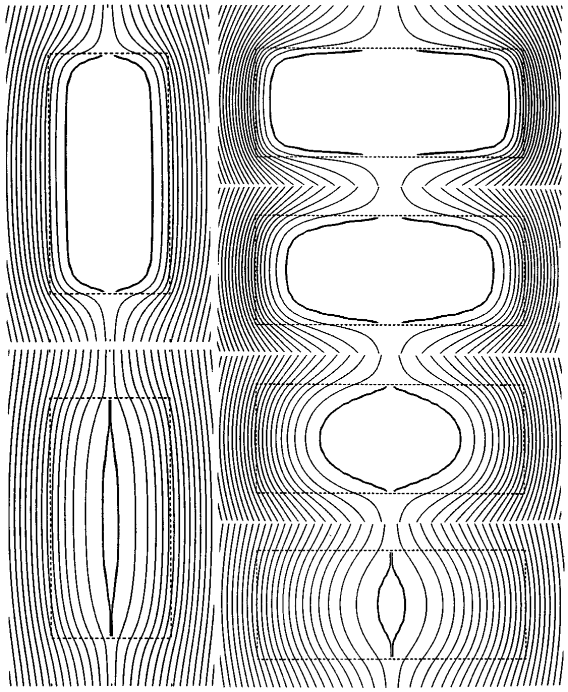

FIG. 1. The field lines of the induction **B** during penetration of perpendicular flux into long bars with side ratios $b/a = 2$ (left) and $b/a = 0.4$ (right) in the Bean limit ($n = 51$) for applied fields $B_a = 0.3, 0.7$ ($B_a = 0.05, 0.1, 0.2, 0.4$, in units $\mu_0 J_c a$. The field of full penetration is $B_p = 0.96$ (0.87, 0.72, 0.49, 0.33) for $b/a = 4$ (2, 1, 0.4, 0.2). The dashed rectangle marks the specimen boundary. The bold line (composed of the merging contour lines $|J|/J_c = 0.5, \ldots, 0.3$) shows the penetrating flux front inside which $J = 0$ and $\mathbf{B} = 0$.

Here and in the following $\mathbf{r}$ denotes (x, y) and the integration is over the specimen cross section S. The induction law $\dot{\mathbf{B}} = \partial \mathbf{B}/\partial t = -\nabla \times \mathbf{E}$ in this geometry reduces to $\dot{A} = -E$. Combining this with Eq. (2) and with the material law $E = E(J)$ or $J = J(E)$, one obtains the equation of motion for $E(\mathbf{r}, t)$,

$$E(\mathbf{r}, t) = \mu_0 \int_S d^2r'\, Q(\mathbf{r}, \mathbf{r}') \frac{\partial}{\partial t} J[E(\mathbf{r}', t)] + x\dot{B}_a . \quad (3)$$

Figure 1 shows the magnetic field lines in and around a rectangular superconductor during flux penetration obtained from a similar integral equation for A.

In creep experiments one has $B_a = \text{const}$; thus $\dot{B}_a = 0$, and Eq. (3) simplifies to

$$E(\mathbf{r}, t) = \mu_0 \int d^2r'\, Q(\mathbf{r}, \mathbf{r}') \dot{E}(\mathbf{r}', t) \frac{\partial J}{\partial E} . \quad (4)$$

This integral equation describes the nonlinear and nonlocal diffusion of $E(\mathbf{r}, t)$ during flux creep. For the power law $E(J)$ (1), one has $\partial J/\partial E = 1/(\partial E/\partial J) = (J_c/\eta E)(E/E_c)^{1/\eta}$, yielding

$$E(\mathbf{r}, t) = \frac{\mu_0 J_c}{n E_c^{1/n}} \int d^2r'\, Q(\mathbf{r}, \mathbf{r}') \dot{E}(\mathbf{r}', t) E^{(1/n)-1} . \quad (4a)$$

Remarkably, this nonlinear integral equation can be solved exactly by the ansatz $E = f(\mathbf{r}) g(t)$, yielding

$$E(\mathbf{r}, t) = E_c f_n(\mathbf{r}) \left(\frac{\tau}{t_1 + t} \right)^{n/(n-1)} . \quad (5)$$

If we choose $\tau = \mu_0 J_c S / 4\pi(n-1) E_c$ we obtain $f_\eta(\mathbf{r})$ from

$$f_n(\mathbf{r}) = -\frac{4\pi}{S} \int d^2r'\, Q(\mathbf{r}, \mathbf{r}') f_n(\mathbf{r}')^{1/n} . \quad (6)$$

This nonlinear integral equation is easily solved by iterating the relation $f_n^{(m+1)} = -(4\pi/S) \int Q f_n^{(m)1/n}$ starting with $f_n^{(0)} = 1$. The resulting series $f_n^{(1)}, f_n^{(2)}, \ldots$, converges rapidly if $n > 1$. For $m \gg 1$ one has approximately $f_n^{(m+1)}(\mathbf{r}) \approx f_n^{(m)}(\mathbf{r})(1 + c/n^m)$ with $c \approx 1$. For $n \gg 1$ (practically for $n \geq 5$) the shape $f_n(\mathbf{r})$ of the electric field becomes a universal function which depends only on the specimen shape but not on the exponent n,

$$f_{n \geq 5}(\mathbf{r}) \approx f_\infty(\mathbf{r}) = -\frac{2}{S} \int_S d^2r' \ln|\mathbf{r} - \mathbf{r}'| . \quad (7)$$

In the Ohmic case $n = 1$, the ansatz (5) makes no sense since the exponent $n/(n-1)$ diverges. For this special case the integral Eq. (4) is linear since the factor $\partial J/\partial E = \sigma = 1/\rho$ is the constant Ohmic conductivity. This linear integral equation is solved by the ansatz

$$E(\mathbf{r}, t) = E_0 f_0(\mathbf{r}) \exp(-t/\tau_0) , \quad (8)$$

yielding the relaxation time $\tau_0 = \mu_0 \sigma S / \Lambda$, where Λ and $f_0(\mathbf{r})$ are the lowest eigenvalue and eigenfunction of the linear integral equation

$$f_0(\mathbf{r}) = -\frac{2\Lambda}{S} \int_S d^2r' \ln|\mathbf{r} - \mathbf{r}'| f_0(\mathbf{r}') . \quad (9)$$

From (1) and (5) the current density becomes

$$J(\mathbf{r}, t) = J_c |f_n(\mathbf{r})|^{1/n} \left(\frac{\tau}{t_1 + t} \right)^{1/(n-1)} \operatorname{sgn} f_n(\mathbf{r}) . \quad (10)$$

For $n \gg 1$, $t \gg t_1$ the time factor in (10) equals $1 - \frac{1}{n-1} \ln(t/\tau)$ and one has $|J| \approx J_c$ everywhere.

The magnetization along y per unit length along z is

$$M(t) = \hat{\mathbf{y}} \mathbf{m}(t)/L_z = \int_S x J(x, y, t)\, dx\, dy . \quad (11)$$

In (11) the prefactor $\frac{1}{2}$ of $\mathbf{m}(t) = \frac{1}{2} \int \mathbf{r} \times \mathbf{J}(\mathbf{r}, t) d^3r$ was compensated by the contribution to $\mathbf{m}$ of the U-turning currents at the far away ends of the bar at $z = \pm L_z/2 \gg a, b$. The resulting factor of 2 in M was sometimes missed in previous work on slabs.

These results apply to arbitrary specimen cross section S. For the realistic rectangular cross section $-a \leq x \leq a$, $-b \leq y \leq b$, $S = 4ab$, with B_a along y, the symmetry $J(x, y) = -J(-x, y) = J(x, -y)$ (and same symmetry for E and A) allows one to restrict the integration to the quarter $0 \leq x \leq a$, $0 \leq y \leq b$ by using the symmetric kernel

$$Q_{\text{sym}}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi} \ln \frac{(x_-^2 + y_-^2)(x_-^2 + y_+^2)}{(x_+^2 + y_-^2)(x_+^2 + y_+^2)} , \quad (12)$$

with $x_\pm = x \pm x'$, $y_\pm = y \pm y'$. One now has in (5)

$$\tau = \mu_0 J_c ab / [\pi(n-1)E_c], \tag{13}$$

$$f_n(x,y) = \int_0^a \frac{dx'}{a} \int_0^b \frac{dy'}{b} \ln \frac{(x_+^2 + y_-^2)(x_+^2 + y_+^2)}{(x_-^2 + y_-^2)(x_-^2 + y_+^2)} \times f_n(x',y')^{1/n}. \tag{14}$$

Before discussing the rectangular cross section I first show that in the limits of longitudinal ($b \gg a$) and transverse ($b \ll a$) geometry the above expressions reduce to known and new results. In both limits, the functions J, E, A, B_x, and B_y depend only on x. One may thus put $y = 0$ writing, e.g., $E(x,y) \approx E(x,0) = E(x)$, and one gets $M = 4b \int_0^a xJ(x)\,dx$.

In the *transverse* limit (strip with thickness $2b \ll a$), the reduced one-dimensional kernel is $Q_{\rm trans}(x,x') = 2bQ_{\rm sym}(x,0,x',0) = (b/\pi)\ln[|x - x'|/(x + x')]$. This kernel was used in calculations of creep [3,10], flux penetration [9], and nonlinear [10] and linear [11,12] ac response of superconducting strips. In the creep solution (5) one now has τ (13) and the shape $f_n(x)$ is determined by

$$f_n(x) = \frac{1}{a}\int_0^a dx' \ln \left| \frac{x + x'}{x - x'} \right| f_n(x')^{1/n}. \tag{15}$$

For $n \gg 1$ this reproduces the E of Ref. [3],

$$E(x,t) = \frac{\mu_0 J_c ab}{\pi(n-1)} \frac{1}{t} \left(\ln \frac{a+x}{a-x} + \frac{x}{a} \ln \frac{a^2 - x^2}{x^2} \right). \tag{16}$$

In the *longitudinal* limit (slab with $b \gg a$), the reduced kernel is $Q_{\rm long}(x,x') = 2\int_0^\infty dy'\, Q_{\rm sym}(x,0,x',y') = \frac{1}{2}(|x - x'| - |x + x'|) = -{\rm Min}(x,x')$ (the minimum value of x, x'), which has the property $Q''_{\rm long} = \delta(x - x')$. One now has in (5) τ from (13) and $f_n(x)$ from

$$f_n(x) = \frac{\pi}{ab} \int_0^a dx' \,{\rm Min}(x,x') f_n(x')^{1/n}, \tag{17}$$

$0 \le x \le a$, $f(-x) = -f(x)$. For large exponents $n \gg 1$ this reproduces the result of Gurevich [1,2],

$$E(x,t) = \frac{\mu_0 J_c}{n-1} \frac{x(2a - |x|)}{2t}. \tag{18}$$

The unphysical dependence of τ (13) and $f_n(x)$ (17), but not of $E(x,t)$ (18), on the specimen length $2b$ may be removed by redefining τ and f_n, multiplying the integrals in (14) and (17) by a factor $1 + \pi b/4a$, and dividing τ (13) by the same factor. This definition leaves the transverse limit unchanged but replaces the prefactor π/ab in (17) by $\pi^2/4a^2$. The integral equation (17) is equivalent to a differential equation with boundary conditions,

$$f_n''(x) = -k^2 f_n(x)^{1/n}, \qquad f_n(0) = f_n'(a) = 0, \tag{19}$$

where $k = \pi/2a$ in our new definition of τ. The solutions of (19) in the Ohmic ($n = 1$) and Bean ($n = \infty$) limits look very similar, $f_1(x) = {\rm const} \times \sin(\pi x/2a)$ and $f_\infty(x) = (\pi^2/8a^2)x(2a - |x|)$, cf. Fig. 2.

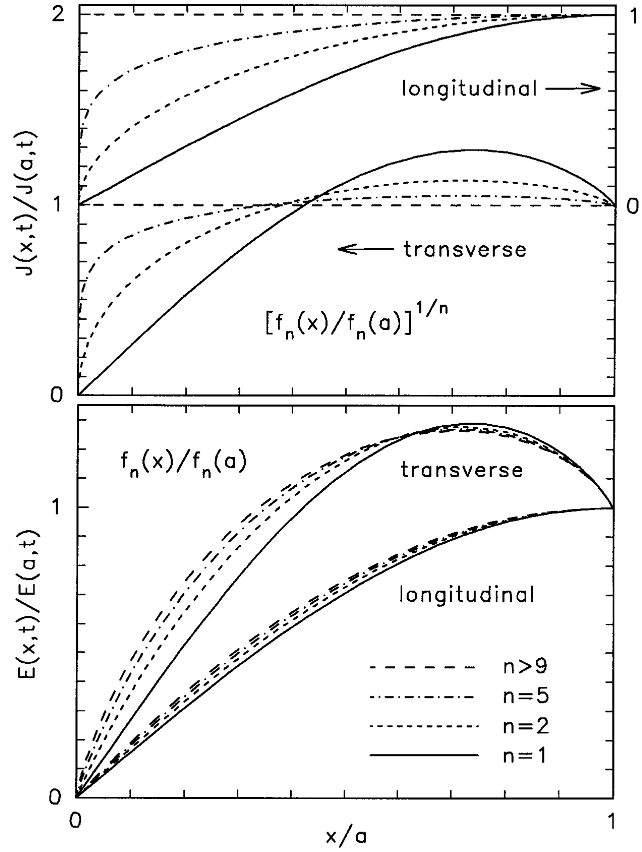

FIG. 2. Profiles of the electric field $E \propto f_n(x)$ and current density $J \propto f_n(x)^{1/n}$ during creep in longitudinal (17) and transverse (15) geometries for exponents $n = 1, 2, 5,$ and ∞.

Creep profiles $f_n(x)$ obtained from (14) and (17) are depicted in Fig. 2 for exponents $n = 1, 2, 5,$ and ∞. Notice that the $f_n(x)$ only weakly depend on n but are qualitatively different for the two geometries: In the longitudinal limit the $f_n(x)$ are maximum at the *boundary* $x = a$, whereas the transverse $f_n(x)$ exhibit a maximum *inside* the specimen, e.g., at $x = 0.735a$ for $n = 1$ [11] and $x = a/\sqrt{2} = 0.707a$ for $n \to \infty$ [3].

The contour lines of the creeping $E(x,y,t)$ inside and outside rectangular bars with aspect ratios $b/a = 2$, 1, and 0.2 are depicted in Fig. 3 for the Bean limit $n \gg 1$. These contour lines of E are also the field lines of the induction $\mathbf{B}(x,y,t)$ *in the remanent state*. Namely, for $n \gg 1$ one gets from (2) and (12) the vector potential

$$A(x,y) \approx -\mu_0 J_c \int_0^a \int_0^b Q_{\rm sym}(\mathbf{r},\mathbf{r}')dy'\,dx' - xB_a, \tag{20}$$

because one has $J \approx J_c\, {\rm sgn} x$ during creep. The lines $A = {\rm const}$ are the field lines of $\mathbf{B} = \nabla \times (\hat{\mathbf{z}}A)$. For $B_a = 0$ these lines coincide with the contour lines of $E = -A$, since during creep the time dependence separates, $E(\mathbf{r},t) = f(\mathbf{r})g(t)$, Eq. (5). For nonzero applied field B_a, the lines $E = {\rm const}$ are still as depicted in Fig. 3. The field lines of $\mathbf{B}$ for B_a near the penetration field B_p look

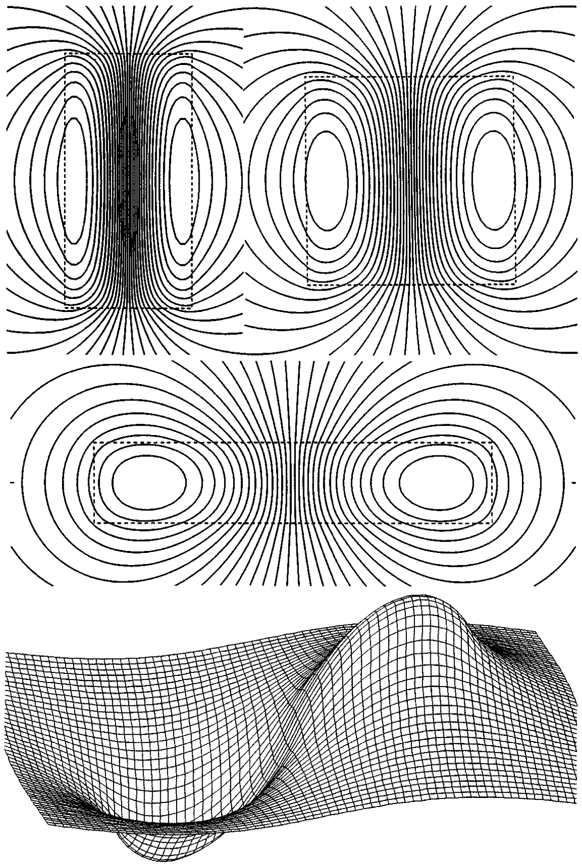

FIG. 3. The contour lines of the electric field $E = -\dot{A}$ during creep in long rectangular superconductors with side ratios $b/a = 2$, 1, and 0.2 (from top) for $n = 51$ and arbitrary constant applied field B_a. These lines coincide with the magnetic field lines in the fully saturated remanent state $B_a = 0$. Bottom: 3D plot of E (and of A) for $b/a = 0.2$.

similar as depicted in Fig. 1, bottom row; they are nearly parallel lines slightly inflated inside the specimen.

All these analytical results are confirmed by computations of creep from Eq. (4): During constant ramp rate $\dot{B}_a \neq 0$ one has $E = x\dot{B}_a$ in the fully penetrated state. When B_a is held constant, the straight E profile starts to curve down near the edge and, after a short transition time t_1, it reaches the universal profiles depicted in Figs. 2 and 3. During creep the *shapes* of E, J, and $\mathbf{B}$ practically do not change over many decades of the creep time t, but their *amplitudes* decrease according to Eqs. (5) and (10). In particular, for $n \gg 1$ and $t \gg t_1$ one has $E \propto 1/t$ and $M \propto J \propto (\tau/t)^{1/(n-1)} \approx 1 - \frac{1}{n-1}\ln(t/\tau)$.

In conclusion, the exact solution for the electric field $E(x,y,t) \parallel z$, and thus for the current density $J \parallel z$ and induction $\mathbf{B} \perp z$, during flux creep is presented for nonlinear conductors with constitutive laws $E \propto J^n$ and $\mathbf{B} = \mu_0\mathbf{H}$ and with rectangular cross section $2a \times 2b$ in a perpedicular field $B_a \parallel y$. The obtained two-dimensional universal profiles of E during creep (Fig. 3) explain how the one-dimensional profiles $E(x,t)$ of the two limiting geometries $b \gg a$ and $b \ll a$ (Fig. 2) come about. In particular, the maximum of $E(x,t)$ in transverse geometry, corresponding to a maximum in the nearly constant $J = J_c(E/E_c)^{1/n} \approx J_c$ and in the energy dissipation $EJ \propto E^{1+1/n} \propto J^{n+1}$, is related to the maximum of the vector potential $A \parallel z$ (2) caused by this current and depicted in Fig. 3. This theory is easily extended to circular disks or cylinders with finite height by modifying the kernel $Q(\mathbf{r},\mathbf{r}')$.

The origin of the well known sign reversal of the perpendicular component B_y at the surfaces $y = \pm b$ near the edges $x = \pm a$, occurring in the remanent state and leading to a neutral line on which $B = B_a$ stays constant during creep [3,5], becomes clear from the magnetic field lines in Fig. 3, which coincide with the contour lines of E and A during creep. Interesting further effects [13] are expected at positions with small B if a more general $B(H)$ with finite B_{c1} is accounted for. Work in this direction is under way.

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