1D Generalized Statistics Gas: A Gauge Theory Approach

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A field theory with generalized statistics in one space dimension is introduced. The statistics enters the scene through the coupling of the matter fields to a statistical gauge field, as it happens in the Chern-Simons theory in two dimensions. We study the particle-hole excitations and show that the long wavelength physics of this model describes a gas obeying the Haldane generalized exclusion statistics. The statistical interaction is found to provide a way to describe the low-T critical properties of one-dimensional non-Fermi-liquids. [S0031-9007(96)00207-4]

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In the Landau theory of Fermi liquids [1] the interaction effects are treated as perturbations and we are introduced to the concept of quasiparticles having a one-to-one correspondence with the single particle states of the ideal Fermi gas. A departure from this Fermi liquid picture can be observed in the study of strongly correlated electron systems, where even for weak interaction the Fermi surface is drastically altered. For many years the theoretical ground for these studies has been the one-dimensional (1D) electron liquid described by the Luttinger model [2,3]. The discovery of the fractional quantum Hall effect (FQHE) [4] in 2D systems gave further momentum to the study of non-Fermi-liquids, since it cannot be explained in terms of single particle states. In fact the edge excitations of a FQHE sample are believed to be described by a 1D non-Fermi-liquid, based on the (chiral) Luttinger model [5].

It was recently proposed [6] that by bosonization of an ideal gas obeying a generalized exclusion statistics [7,8] we may describe the low-T fixed points of 1D non-Fermi-liquids. Instead of being based on the monodromy properties of the wave functions, this generalized exclusion statistics is based on the variation of the number of available single particle states as the number of particles in the system varies through the relation $\Delta d = -\kappa \Delta N$, with d being the number of available states, N the number of particles, and κ the "statistical interaction" parameter. For $\kappa = 0$ we have bosons and for $\kappa = 1$ we have fermions, and for other values we say that we are dealing with generalized statistics. In this Letter we propose another point of view for this correspondence between generalized statistics and the Luttinger model. Inspired by the success of the Chern-Simons-Ginzburg-Landau description for the FQHE in 2D [9], where the (spinless) electrons are described by bosonic fields coupled to a Chern-Simons gauge field in a way that the effective theory is fermionic, we recently introduced [10] a 1D gauge field theory that when coupled to the matter fields has the property of transmuting the statistics of the elementary quanta. Using this gauge model we here explore the long wavelength physics and find that for the particle-hole (density) fluctuations, the statistical parameter κ can be used to relate the density-density correlation function of this model to the ones found for the 1D Thirring and Luttinger models.

We now proceed to describe our gauge model. Imagine a gas of bosonic and spinless nonrelativistic particles, constrained to move on an infinite line and that in the many-body language are described as the elementary quanta of a complex matter field $\Psi(x,t)$. The only interaction present here is a "statistical" gauge interaction. The Lagrangian density for this system is

$$\mathcal{L} = i\Psi^*(\partial_t + i\phi)\Psi + \frac{1}{2m}\Psi^*\left(\partial_x - i\partial_x\xi + \frac{i}{2}\chi\right)^2 \times \Psi - \frac{1}{2\kappa\pi}(\chi\partial_t\xi + \phi\chi), \tag{1}$$

with κ a real parameter. It is easy to see that this Lagrangian density is invariant under the gauge transformations

$$\Psi'(x,t) = \Psi(x,t)e^{i\Lambda(x,t)},$$
$$\Psi'^*(x,t) = \Psi^*(x,t)e^{-i\Lambda(x,t)}, \tag{2}$$

$$\xi'(x,t) = \xi(x,t) + \Lambda(x,t), \quad \chi'(x,t) = \chi(x,t), \tag{3}$$

$$\phi'(x,t) = \phi(x,t) - \partial_t\Lambda(x,t). \tag{4}$$

The two scalar gauge fields ξ and χ enter the x component of the covariant derivative in a combination that transforms as a vector potential. The last term in $\mathcal{L}$ gives the dynamics of the statistical gauge fields.

To quantize the above model we follow [11] where a gauge invariant treatment of the two-dimensional Chern-Simons model can be found. From the symplectic structure of (1) we have the following equal time canonical commutation relations (nonzero part):

$$[\hat\Psi(x,t), \hat\Psi^\dagger(y,t)] = \delta(x-y),$$
$$[\hat\xi(x,t), \hat\chi(y,t)] = -i2\kappa\pi\delta(x-y). \tag{5}$$

The Hamiltonian operator is given by

$$\hat{H} = \int_{-\infty}^{\infty} dx \left\{ -\frac{1}{2m} \hat{\Psi}^\dagger \left[\partial_x - i \left(\partial_x \hat{\xi} - \frac{1}{2} \hat{\chi} \right) \right]^2 \times \hat{\Psi} + \hat{\phi} \left(\frac{1}{2\kappa\pi} \hat{\chi} + \hat{\Psi}^\dagger \hat{\Psi} \right) \right\}. \quad (6)$$

As in electrodynamics the term that multiplies $\hat{\phi}$ in $\hat{H}$ is the generator of time independent gauge transformations and can be set identically zero if we work with gauge invariant matter fields, i.e., fields that commute with $\hat{G} = \frac{1}{2\kappa\pi}\hat{\chi} + \hat{\Psi}^\dagger\hat{\Psi}$. These fields are given by the gauge invariant operators

$$\hat{\Phi}(x,t) = \hat{\Psi}(x,t) e^{-i\hat{\xi}(x,t)}, \qquad \hat{\Phi}^\dagger(x,t) = \hat{\Psi}^\dagger(x,t) e^{i\hat{\xi}(x,t)}. \quad (7)$$

With these operators and setting $\hat{G} = 0$, the Hamiltonian reads

$$\hat{H} = -\int_{-\infty}^{\infty} dx \frac{1}{2m} \hat{\Phi}^\dagger (\partial_x - i\kappa\pi \hat{\Phi}^\dagger \hat{\Phi})^2 \hat{\Phi}. \quad (8)$$

To study the content of the above $\hat{H}$ we introduce the occupation number operator

$$\hat{N} = \int_{-\infty}^{\infty} dx\, \hat{\Phi}^\dagger \hat{\Phi} \quad (9)$$

and construct the following arbitrary eigenstate of $\hat{N}$:

$$|N\rangle = \int dx_1 \cdots dx_N\, \psi(x_1, \ldots, x_N) \times \hat{\Phi}^\dagger(x_1) \cdots \hat{\Phi}^\dagger(x_N) |0\rangle, \quad (10)$$

with ψ being an arbitrary function symmetric under exchange of any of its entries and $|0\rangle$ is the ground state defined by $\hat{\Phi}(x)|0\rangle = 0$. If we further insist that this state is an eigenstate of $\hat{H}$ with eigenvalue E, we have that ψ satisfies the N-body Schrödinger equation

$$-\frac{1}{2m} \sum_{a=1}^{N} \left[\frac{\partial}{\partial x_a} - i\kappa\pi \sum_{b \neq a} \delta(x_a - x_b) \right]^2 \times \psi(x_1, \ldots, x_N) = E\psi(x_1, \ldots, x_N). \quad (11)$$

This δ-function interaction was considered in [12] for the particular case $\kappa = 1$ in order to describe fermions in the Feynman path integral. One can remove this interaction with the aid of a gauge transformation in ψ:

$$\bar{\psi}(x_1, \ldots, x_N) = \exp\left(-i\kappa\pi \sum_{a<b} \theta_H(x_a - x_b) \right) \times \psi(x_1, \ldots, x_N), \quad (12)$$

where θ_H is the Heaviside step function, so that $\bar{\psi}$ satisfies the free Schrödinger equation

$$-\frac{1}{2m} \sum_{a=1}^{N} \frac{\partial^2}{\partial x_a^2} \bar{\psi}(x_1, \ldots, x_N) = E\bar{\psi}(x_1, \ldots, x_N), \quad (13)$$

but obeys a nontrivial condition under exchange of any two arguments

$$\bar{\psi}(x_1, \ldots, x_a, x_b, \ldots, x_N) = e^{i\kappa\pi\, \mathrm{sgn}(x_a - x_b)} \times \bar{\psi}(x_1, \ldots, x_b, x_a, \ldots, x_N). \quad (14)$$

As we can see from (14) our model leads to an effective theory where the elementary quanta display a generalized statistics, but for a statistics other than Fermi or Bose, due to strong correlation we will have a situation similar to the Thirring model were there is, asymptotically, no single particle interpretation [13]. This point will become clear in the following, when we study the collective excitations of our model and show its equivalence to the Thirring model.

Within this model we now proceed to study the behavior of the collective modes corresponding to the density fluctuations (particle-hole excitations) of a 1D gas with generalized statistics. For that purpose we decompose the matter field Ψ in a density and phase parts

$$\Psi = \sqrt{\rho}\, e^{i\eta}, \quad (15)$$

and introduce a uniform background density $-\bar{\rho}$, so that (1) reads

$$\mathcal{L} = -\rho\partial_t\eta - (\rho - \bar{\rho})\phi - \frac{1}{2m} \left[(\partial_x\sqrt{\rho})^2 + \rho\left(\partial_x\eta - \partial_x\xi + \frac{1}{2}\chi \right)^2 \right] - \frac{1}{2\kappa\pi}\chi(\phi + \partial_t\xi). \quad (16)$$

Variation of ϕ gives the constraint $\chi = -2\kappa\pi(\rho - \bar{\rho})$. Introducing the variables

$$\sigma = \eta - \xi, \qquad \tau = \eta + \xi, \quad (17)$$

we have that (16) goes to

$$\mathcal{L} = -\rho\partial_t\sigma - \frac{1}{2m}\{(\partial_x\sqrt{\rho})^2 + \rho[\partial_x\sigma - \kappa\pi(\rho - \bar{\rho})]^2\}. \quad (18)$$

Notice that the above Lagrangian depends only on the gauge invariant fields σ and ρ. We now make one more change of variables,

$$\rho = \bar{\rho} + \partial_x\theta, \quad (19)$$

so that the density fluctuations $\rho - \bar{\rho}$ are now expressed as $\partial_x\theta$. As one can see from the first term in (18) $\Pi_\theta = -\partial_x\sigma$ is the canonical momentum of θ in these new variables. If we go to Fourier modes $\theta(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dx\, \theta(x)\exp(iqx)$, the long wavelength physics is governed by the Hamiltonian

$$H \simeq \frac{\bar{\rho}}{2m}\int_{-\infty}^{+\infty} dq[\Pi_\theta(q)\Pi_\theta(-q) + q^2\kappa^2\pi^2\theta(q)\theta(-q)]. \quad (20)$$

Performing the canonical transformation

$$\theta \to \frac{1}{\sqrt{\kappa\pi}}\theta, \quad (21)$$

$$\Pi_\theta \to \sqrt{\kappa\pi}\,\Pi_\theta\,, \tag{22}$$

the Hamiltonian reads

$$H \simeq \frac{v_s}{2}\int_{-\infty}^{+\infty} dq[\Pi_\theta(q)\Pi_\theta(-q) + q^2\theta(q)\theta(-q)]\,, \tag{23}$$

with v_s being the sound velocity

$$v_s = \kappa\frac{\bar\rho\pi}{m} = \kappa v_F\,. \tag{24}$$

Here v_F is the Fermi velocity for a gas of 1D spinless electrons. The same result was found in [6] by bosonization of an ideal gas of particles obeying the Haldane generalized exclusion statistics.

To see the relation of our model with 1D fermion models we now proceed to compute the ground-state wave functional ψ_0 for (23); we choose the wave functional to depend on θ so that Π_θ acts on it as the functional derivative $-i\frac{\delta}{\delta\theta}$. It can be shown [14] that $|\psi_0|^2$ is equivalent to the $2N$-point density-density correlation function provided we set $\partial_x\theta(x)$ to represent the density for N particles at x_a and N holes at y_a ($a = 1, 2, \ldots, N$), i.e.,

$$\begin{aligned}\theta(x) &= \sqrt{\kappa\pi}\sum_{a=1}^{N}[\theta_H(x - x_a) - \theta_H(x - y_a)] \\ &= \sqrt{\kappa\pi}\sum_{a=1}^{N}\frac{1}{2\pi i}\int_{-\infty}^{+\infty} dq\,\frac{1}{q}\,(e^{iq(x-x_a)} - e^{iq(x-y_a)})\,.\end{aligned} \tag{25}$$

The solution for the ground-state wave functional is

$$\psi_0[\theta] = \mathcal{N} e^{(1/2)\int_{-\infty}^{+\infty} dq|q|\theta(q)\theta(-q)}, \tag{26}$$

with $\mathcal{N}$ a normalization constant. If we now extract from (25) $\theta(q)$ and insert it in (26), we obtain

$$|\psi_0|^2 = \mathcal{N}^2\frac{\prod_{a<b}|x_a - x_b|^{2\kappa}|y_a - y_b|^{2\kappa}}{\prod_{a,b}|x_a - y_b|^{2\kappa}}\,. \tag{27}$$

For $\kappa = 1$ we recover the $2N$-point density-density correlation function for free gapless 1D Dirac fermions [15], and for κ generic (but >0) we have the equivalence with the gapless Thirring and Luttinger models [14]. In fact, the use of generalized statistics to solve the Thirring model can be found in [13].

To study the nonideal gas case, one can introduce a two-body interaction in (20) through the term

$$\begin{aligned}H_{\rm int} &= \frac{1}{2}\int dx\int dy[\rho(x) - \bar\rho]V(x - y)[\rho(y) - \bar\rho] \\ &= \sqrt{\frac{\pi}{2}}\int_{-\infty}^{+\infty} dq\,V(q)q^2\theta(q)\theta(-q)\,.\end{aligned} \tag{28}$$

If the two-body potential is sufficiently short ranged, so that in the low q approximation we keep only the term $V(q = 0)$, the sole effect of this interaction in all that we have computed is to renormalize the statistical parameter κ:

$$\kappa \to \sqrt{\kappa^2 + \frac{m\sqrt{2}}{\bar\rho\pi^{3/2}}V(0)}\,. \tag{29}$$

For more general forms of $V(x - y)$ we can build a perturbation theory based on the Luttinger model as the Fermi liquid is based on the ideal Fermi gas [3].

In summary, we described here an alternative description of generalized statistics in 1D based on a gauge field theory that parallels the Chern-Simons construction in 2D. It was shown that in the long wavelength limit we have a correspondence to a gas with generalized exclusion statistics, and the relation to 1D fermionic models was established.

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