Landau-Ginzburg Theory of Self-Organized Criticality

L. Gil[1] and D. Sornette[2]
[1]*Institut Non-Linéaire Niçois, Université de Nice-Sophia Antipolis, 1361 Route des Lucioles, 06560 Valbonne, France*
[2]*Laboratoire de Physique de la Matière Condensée, Université de Nice-Sophia Antipolis, B.P. 70, Parc Valrose, 06108 Nice Cedex 2, France*
(Received 25 August 1995)

Self-organized criticality occurs through a nonlinear feedback mechanism triggering transitions between different *metastable* states. These transitions take the form of intermittent avalanchelike events distributed according to a power law. We present the first and simplest *fully continuous* partial differential formalism of this phenomenon, based on the introduction of a *subcritical* dynamics. SOC is identified as the regime where diffusive relaxation is faster than the instability growth rate. In the other limit of slow diffusion, avalanches comparable to the system size become dominant. This provides a general correspondence between SOC and synchronization of threshold oscillators. [S0031-9007(96)00101-9]

PACS numbers: 64.60.Ht, 05.40.+j, 05.45.+b, 05.70.Ln

The possibility for driven dissipative extended systems to exhibit a spontaneous organization towards a kind of dynamical critical point has been dubbed self-organized criticality (SOC) [1]. This concept has been mostly illustrated using cellular automata models [2,3] and discrete space-time models [4]. They are characterized by a very slow driving and a threshold dynamics, i.e., a local *stepwise* unstability occurs when the field exceeds some critical value leading to a rapid relaxation on neighbors which may cascade to create large avalanches well differentiated in time (this is where the slow driving is important). Attempts have been pursued to develop continuous field theoretical approaches to this phenomenon, based on continuous anisotropic nonlinear driven diffusion equations with stochastic noise [5]. However, avalanches are not described and the origin of the self-organization is not explained. This is due to the fact that the threshold dynamics [6] is replaced by a "weak" perturbative nonlinear term. Furthermore, the driving occurs on a fast time scale (stochastic noise) in contrast with the very slow driving common to all SOC models, whereas the order parameter exhibits slow diffusionlike relaxations similar to critical slowing down [7] in opposition to the fast relaxation induced by the avalanches. A physical system which exemplifies these features is provided by earthquakes which relax, over time scales of tens of seconds, the stress accumulated over centuries.

Our goal is to construct a fully continuous formulation in terms of partial differential equations, in the spirit of the Landau-Ginzburg theory of phase transitions, which takes full account of the nonperturbative nature of the threshold mechanism. Our hope is that a continuous field theory constructed on the basis of symmetry and parsimony will be both sufficiently simple and general to teach us something on SOC, as for thermal critical transitions. We do not describe a specific experimental system but rather aim at a general understanding that will provide a starting point for specific applications. However, for the sake of pedagogy, we formulate the problem in the *sandpile* language. It will turn out that, notwithstanding our neglecting of many details, the basic properties as well as the known value of the avalanche exponent are recovered quantitatively by our approach, providing a new test of universality for SOC as well as a connection with synchronization behavior.

Our first idea is that the theory must incorporate both the dynamics of an order parameter (OP) and of the corresponding *control* parameter (CP), in order to understand why the CP self-organizes to a critical value. Within the sandpile picture $\partial h/\partial x$ is the slope of the sandpile, h being the local height, and S is the state variable distinguishing between static grains ($S = 0$) and rolling grains ($S \neq 0$). Therefore, the sand flux is proportional to S. Coupling these two parameters is very natural physically and has already been exploited to describe the large avalanche regime rotating drum experiments [8,9].

Our second ingredient is to specify the dynamical equation of the order parameter. The crucial role played by the threshold dynamics and the necessity to take it into account explicitly in a continuum formalism has been recognized by several authors [6,10–13]. In these works, the threshold nature of the dynamics is modeled by either a discontinuous or singular diffusion coefficient [10,12,13] or by series expansion of the Heaviside function and its derivatives [11]. Therefore, their formalism still contains an *ad hoc* discrete component. Furthermore, the very slow driving condition is rarely imposed except in [13]. The only attempt to incorporate the threshold behavior in a continuous formalism has been done in [9] using the macroscopic phenomenological Coulomb solid friction law. It turns out that this law does not yield any SOC but only a large avalanche regime due to the linear growth derived from the Coulomb law. Here we propose a more microscopic and fundamental description which can both display SOC and the large avalanche regime, and

0031-9007/96/76(21)/3991(4)$10.00

furthermore provides a good model of solid friction [14]. Our basic idea is that the OP presents multistability and hysteresis. This will ensure the threshold dynamics as well as all the other properties described below. In addition, it can be shown that a local hysteretic response qualifies as a microscopic model of solid friction [14], which is a key property of dry sand. This justifies further our choice to incorporate in the theory this fundamental two-state property. The simplest way to implement this condition in a continuous formalism is to write down the normal form for a *subcritical* bifurcation, characterized by the possible coexistence of two local states:

$$\frac{\partial S}{\partial t} = \chi\{\mu S + 2\beta S^3 - S^5\}, \qquad (1)$$

where $\mu = [(\partial h/\partial x)^2 - (\partial h/\partial x|_c)^2]$ and $\beta > 0$ (subcritical condition). In the supercritical case ($\beta < 0$), one can show that Eq. (1) together with the feedback mechanism given by Eq. (2) below reduces to a simple diffusive relaxation of the slope with small fluctuations without avalanches [15]. $\partial h/\partial x|_c$ is the critical slope beyond which the sand begins to flow. The parameter χ fixes the time scale ($\sim\chi^{-1}$) for the growth of S once an instability sets in. Odd powers of S in Eq. (1) ensure that the physics is invariant with respect to $S \to -S$ since the sand flux should always be downslope (see below). For the time being, consider μ as a fixed parameter. Expression (1) ensures that the value of S^2 which is selected at long times is not a continuous function of μ, as expected for a hysteresis or first-order behavior. Indeed, when μ is given, Eq. (1) is variational: $\chi^{-1}\partial S/\partial t = -\delta\mathcal{F}/\partial S$, which implies that the asymptotic solution for S is one of the states which minimizes $\mathcal{F} = -(\frac{1}{2}\mu S^2 + \frac{1}{2}\beta S^4 - \frac{1}{6}S^6)$ given by $S_I = 0$ and $S_{\rm II}^2 = \beta + \sqrt{\beta^2 + \mu}$ ($S_{\rm III}^2 = \beta - \sqrt{\beta^2 + \mu}$ is an unstable fixed point). These two states have the same energy $\mathcal{F}(S_{\rm I}) = \mathcal{F}(S_{\rm II})$ at $\mu = \mu_e = -\frac{3}{4}\beta^2$. Suppose now that the local slope $\partial h/\partial x$ increases steadily to larger and larger negative values (we consider the case of a sandpile whose slope decreases on average from left to right). The $S = 0$ state remains stable until $\partial h/\partial x$ reaches $-\sqrt{(\partial h/\partial x|_c)^2 + \mu_e}$. Beyond this slope, the $S = 0$ state becomes metastable to the $S = S_{\rm II}$ state. However, in the absence of large fluctuations enabling to pass the barrier, the OP will remain zero until the slope reaches the "spinodal" value $\partial h/\partial x|_c$ ($\mu = 0$). At this point, Eq. (1) shows that $S = 0$ is linearly unstable (μ becomes positive) as $S = S_0 e^{\mu\chi t}$ at short times (S_0 being some small initial fluctuation) and $S \to S_{\rm II}$ at long times. The physical mechanism for this unstability can be tracked back to the unstable branch $S_{\rm III}^2$ which describes an increase of the OP and therefore flux [see Eq. (3) below] when the slope decreases in absolute value.

The last ingredient of the theory is the feedback of the OP S onto the CP ($\partial h/\partial x$ or μ) which becomes a dynamical variable and relaxes by the transition to the dynamical state $S \neq 0$ [16]. Physically, when a grain starts to roll down the slope, this tends to relax the slope which has led to its instability. However, any feedback mechanism leading to the organization of the system close to the critical state is not able to give rise to long-range order and power-law distributions (explicit examples can be built [15]). One has also to take into account the physical fact that a local relaxation of the *CP* takes place at the expense of the neighbors, a local relaxation of the slope being associated with an increase of the slope of the neighbors. A sufficient but not necessary condition for this is, and that will be our last condition, that a conservation law operates on the control parameter [17]. We adopt it having in mind the physical conservation of grains in a sandpile or of stress in the earthquake problem. It is expressed as

$$\frac{\partial h}{\partial t} = -\frac{\partial F(S, \partial h/\partial x)}{\partial x} + \Phi\,, \qquad (2)$$

where F is a (grain) flux and Φ is a weak, slow random source term which describes the continuous driving of the system. Symmetry considerations now allow us to get the form of $F(S, \partial h/\partial x)$: $F(0, \partial h/\partial x) = 0$ (there is no flux if there are no rolling grains); $F(-S, \partial h/\partial x) = F(S, \partial h/\partial x)$ (the sign of S has no meaning); $F(S, -\partial h/\partial x) = -F(S, \partial h/\partial x)\,[(x \to -x) \to (F \to -F)\,(\text{parity})]$; $\partial h/\partial x > 0 \to F(S, \partial h/\partial x) < 0$ (the sand falls down the slope). The simplest expression obeying these criteria is

$$F\left(S, \frac{\partial h}{\partial x}\right) = -\alpha\,\frac{\partial h}{\partial x}\,S^2, \qquad \alpha > 0\,. \qquad (3)$$

This expression (3) together with (2) determine the feedback of the OP S onto the CP h (or better $\partial h/\partial x$) capturing the obvious fact that stress (slope) may be relaxed only on active $S \neq 0$ sites. Here we neglect higher order terms in (3).

Now we analyze the set of coupled Eqs. (1)–(3). In the language of dynamical systems, one can represent the dynamics at a given spatial position x along the system and given time t as a representative point (RP) in the phase space {CP, OP}. Because of the external forcing and transfer of h from active neighbors, the RP moves along the $S_{\rm I}$ branch and eventually reaches the spinodal point $\mu = 0$. Then, two main regimes occur depending on the value of the key parameter χ/α. Consider first the case $\chi/\alpha \gg 1$ for which the dynamics of the OP is much faster that of the diffusive relaxation of the CP. In the presence of some background noise, the limit $\chi \to +\infty$ leads to a very fast jump of the OP from the state of repose $S = 0$ to the active state $S = S_{\rm II}$. Because of the diffusive relaxation (2), the RP then follows the upper branch downward to lower absolute values of the CP. This will hold down to the limiting slope $-\sqrt{(\partial h/\partial x|_c)^2 - \beta^2}$, at which the $S_{\rm II}$ and $S_{\rm III}$ branches coincide and beyond which the only solution for the OP is $S = 0$. At this point, the RP has to jump back to the lower stable branch and the avalanche has gone past

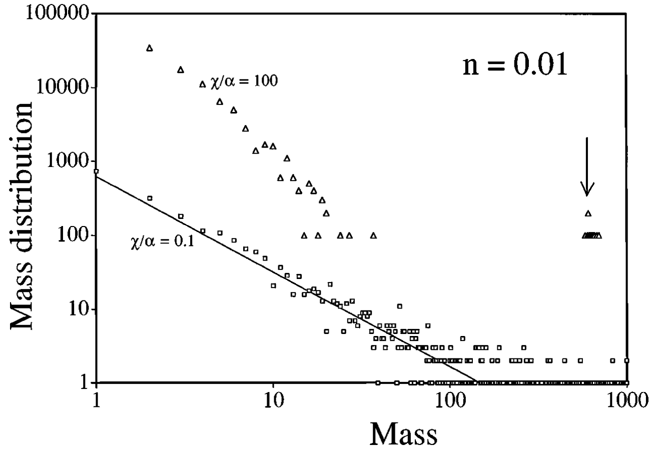

FIG. 1. This figure shows the distributions $P(M)$ of the avalanche sizes for the two regimes: $\chi/\alpha = 0.1$ and $\chi/\alpha = 100$. The curves have been moved with respect to each other for better clarity. We define an avalanche as the loci of connected points whose local flux is larger than some threshold, here $\beta/100$. We measure the mass of an avalanche as the integral over time of the flux going out of the system on its right, which corresponds to actual experiments [21]. We have found convenient to add a small noise source n on the rhs. of Eq. (1) in order to help start up the instabilities. Most of the simulations have been done with a time step of integration equal to 10^{-3} which is much smaller than the characteristic time scale $\chi^{-1} = 1$, while we have verified that changing it by 1 order of magnitude on both sides does not modify our results. The driving is done by choosing at random the position at each time step at which an increment of flux is added locally. The boundary conditions are $\partial h/\partial x|_{x=0} = 0$ and $h_{x=L} = 0$. System sizes range from $L/a = 64$ to 2048. The parameters are spinodal slope $|\partial h/\partial x_c| = 1.2$, driving flux $\Phi = 0.1$, subcritical parameter $\beta = 1.5$.

this point. Because of the slow continuous forcing, this hysteresis cycle repeats itself, leading to almost periodic large scale avalanches (see Fig. 1). Alternatively, one can see the set of equations (1)–(3) as reducing to the original sandpile cellular automaton rules for $\chi \to +\infty$ which in 1D do give rise to repetitive avalanches which span the whole system in a dominolike pattern [1]. This regime is reminiscent of real sandpiles that are well documented to exhibit large quasiregular avalanches corresponding to oscillations of relaxation between two different angles of repose [9,18], the maximum angle $|\partial h/\partial x|_c|$ for triggering a dynamical flow and the minimum angle $-\sqrt{(\partial h/\partial x|_c)^2 - \beta^2}$ which can still sustain a sand flow [19]. Physically, we suggest that this regime stems from the fact that the halting of a rolling grain occurs over a time which is comparable or smaller than the time needed for a significant decrease of the local slope by diffusion, hence the condition $\chi/\alpha \gg 1$.

In the other regime χ/α not large, the time scale χ^{-1} over which the OP grows is now comparable or larger than the smallest relaxation time scale due to the diffusive relaxation of the CP: The OP has no time to jump to the upper branch before it is perturbed by the diffusion of the CP. In this regime, one can approximate the trajectory of the RP in phase space close to the spinodal point by $S^2 = -\mu/2B$, where $0 < B < \beta$ (the upper bound corresponds to a trajectory which is tangent to the unstable branch S_{III}). In the vicinity of the spinodal point, we report this expression for S^2 into (2) and (3) and get an *antidiffusion* equation for h with a *negative* diffusion coefficient $D = -(\alpha/B)(\partial h/\partial x|_c)^2$. The growth of a given mode h_q is then given by $h_q(t) \simeq e^{|D|q^2 t}$ and is all the more pronounced for small wavelengths. In practice, a finite mesh size a or higher order terms will provide an ultraviolet cutoff. This mechanism corresponds to a cascade from small scales to large scales which is very sensitive to microscopic heterogeneities and leads to the broad power law distribution of avalanches (see Fig. 1). This is analogous to the inverse energy cascade in 2D turbulence and is reminiscent of the inverse cascade in the Kuramoto-Sivashinsky equation leading to scale invariant solutions described by the KPZ equation [20].

These predictions have been checked by a numerical implementation of the equations of motions (1)–(3). Figure 1 presents the distributions $P(M)$ of avalanche sizes M for the two regimes: For $\chi/\alpha = 0.1$, the distribution qualifies as a power law $P(M)dM \simeq M^{-(1+\mu)}dM$, with an exponent $\mu = 1.0 \pm 0.1$ which is found independent of χ/α in this regime which extends up to about $\chi/\alpha = 1$. This value is compatible with experiments: $\mu \simeq 1.5$ in Ref. [21] and $\simeq$1.1 in Ref. [22]. The avalanche mass distributions for different system sizes L/α obey finite size scaling as they collapse on the same master curve using the reduced variable M/L^{σ}, with $\sigma = 0.7 \pm 0.1$. For $\chi/\alpha > 1$, we still observe a power-law distribution at small avalanche sizes with exponents μ increasing continuously with χ/α. However, the dominant structure is the appearance of very large avalanches spanning the whole system represented on the plot by the peak, in agreement with our above analysis.

Figure 2 shows a subtle effect which has also been observed in experiments [23], namely the effect of the noise n added on the OP. Figure 2 represents the avalanche mass distribution for the same $\chi/\alpha = 0.1$ but decreasing values of the noise n. While the mass distributions remain a power law for small masses with the same exponent μ, decreasing the noise introduces a characteristic scale: Because of the small albeit finite driving (present here for numerical feasibility and corresponding to the discrete nature of grains in experiments), if the noise is too small, the spinodal point $\mu = 0$ can be bypassed locally ($\mu > 0$) and as a consequence the Lyapunov exponent $\chi\mu$ becomes finite leading to a jump to the upper branch S_{II} even in this regime $\chi/\alpha < 1$ for which the upper branch should be in principle never attained for sufficiently slow driving.

Figure 3 shows the distribution $P(J)$ of flux amplitudes at the right border, where the flux is going out of the system in the same condition as for Fig. 1. For $\chi/\alpha = 0.1$ (and all other values less than about 1), $P(J)dJ \simeq M^{-\delta}dJ$, with $\delta = 0.7 \pm 0.1$ independent from χ/α in this regime. This power law describes the *small*

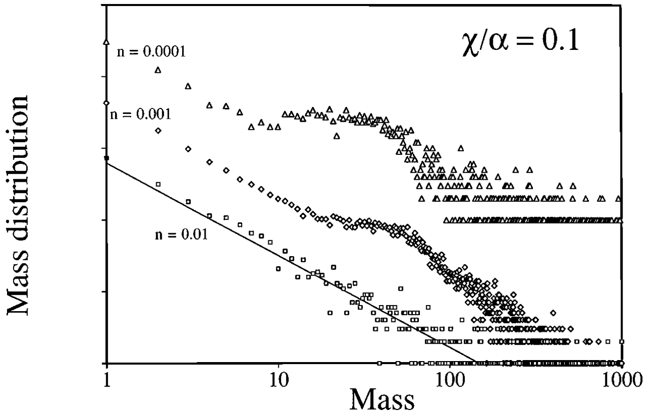

FIG. 2. Distributions $P(M)$ of avalanche sizes for the same $\chi/\alpha - 0.1$ but decreasing values, from bottom to top, of the noise. The curves have been moved with respect to each other for better clarity.

$J \rightarrow 0$ flux distribution, as J is bound by the intrinsic value $J_{\max} = \alpha S_{11}^2 \partial h/\partial x|_c$. The mass M is related to the avalanche duration T and the statistics of flux by $M = \int_0^T J(t)\,dt \simeq T\langle J\rangle_T$. Since $\delta < 1, \langle J\rangle \sim J_{\max}^{2-\delta}$ is constant and the statistics of M is the same as that of the duration T, as we checked directly. This also agrees with experiments. For $\chi/\alpha > 1$, we observe even more clearly than on Fig. 1 the existence of a characteristic flux scale, whose size is simply given by $J_{\max}$.

In summary, since first-order (subcritical) behavior is much more common than (super) critical transitions in nature, our theory suggest an explanation for the genericity of SOC, via the spatiotemporal coupling of first-order transitions. The remarkable outcome is that this coupling of oscillators of relaxation produces a spontaneous organization, under suitable conditions, characterized by scale invariant avalanches.

We thank J.-L. Meunier and M. Mendjeli for stimulating discussions and the Conseil Général des Alpes Maritimes for financial support. INLN (resp. LPMC) is UMR 129 (resp. URA 190) of the French National Center for Scientific Research (CNRS).

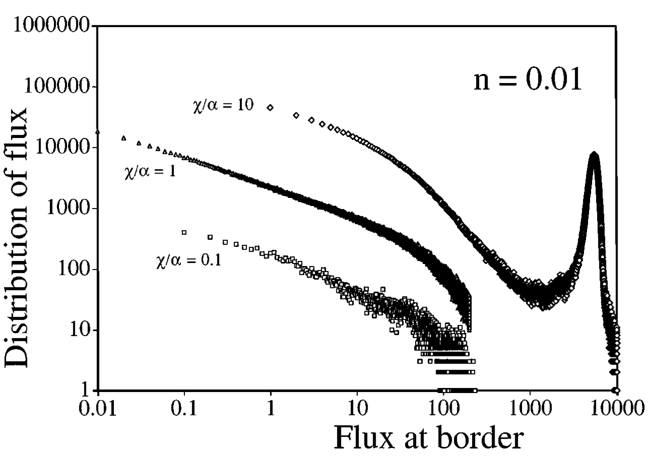

FIG. 3. Distribution $P(J)$ of flux amplitudes at the right border, in the same conditions as for Fig. 1.

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).
[2] D. Dhar and R. Ramaswamy, Phys. Rev. Lett. **63**, 1659 (1989); D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990); Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992); K. Christensen and Z. Olami, Phys. Rev. A **46**, 1829 (1992).
[3] K. Christensen, Ph.D. Thesis, Oslo, 1992; A. Corral, C.J. Pérez, A. Díaz-Guilera, and A. Arenas, Phys. Rev. Lett. **74**, 118–121 (1995); A.A. Middleton and C. Tang, Phys. Rev. Lett. **74**, 742–745 (1995); S. Bottani, Phys. Rev. Lett. **74**, 4189 (1995).
[4] P. Bak and K. Sneppen, Phys. Rev. Lett. **71**, 4083 (1993); P. Miltenberger, D. Sornette, and C. Vanneste, Phys. Rev. Lett. **71**, 3604 (1993); H. Flyvbjerg, K. Sneppen, and P. Bak, Phys. Rev. Lett. **71**, 4087 (1993); T.S. Ray and N. Jan, Phys. Rev. Lett. **72**, 4045 (1994).
[5] T. Hwa and M. Kardar, Phys. Rev. A **45**, 7002 (1992); P.-M. Lam and F. Family, Phys. Rev. E **47**, 1595 (1993); G. Grinstein and D.-H. Lee, Phys. Rev. Lett. **66**, 177 (1991); G. Grinstein, J. Appl. Phys. **69**, 5441 (1991); J. Krug, J.E.S. Socolar, and G. Grinstein, Phys. Rev. A **46**, R4479 (1992).
[6] D. Sornette, *Images de la Physique 1993* (CNRS, Paris, 1994), édition du CNRS.
[7] P.C. Hohenberg and B.I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).
[8] J.-P. Bouchaud, M.E. Cates, J. Ravi Prakash, and S.F. Edwards, J. Phys. I (France) **4**, 1383 (1994).
[9] C.A. Caponeri, S. Douady, S. Fauve, and C. Laroche, in *Mobile Particulate Systems,* edited by E. Guazzelli and L. Oger, NATO ASI Series 287, Vol. 19 (Kluwer, Dordrecht, The Netherlands, 1995).
[10] P. Bántay and I.M. Jánosi, Phys. Rev. Lett. **68**, 2058 (1992).
[11] A. Díaz-Guilera, Phys. Rev. A **45**, 8551 (1992); Fractals **1**, 963 (1993).
[12] R. Cafiero, V. Loreto, L. Pietronero, A. Vespignani, and S. Zapperi, Europhys. Lett. **29**, 111 (1995).
[13] E.T. Lu, Phys. Rev. Lett. **74**, 2511 (1995).
[14] C. Caroli and P. Nozières, in *The Physics of Friction,* edited by B. Peisson (to be published).
[15] L. Gil and D. Sornette (to be published).
[16] D. Sornette, J. Phys. I (France) **2**, 2065 (1992); N. Fraysse, A. Sornette, and D. Sornette, J. Phys. I (France) **3**, 1377 (1993).
[17] We shall not explore here other possibilities where the coupling is nonconservative; see, however, [15].
[18] P. Evesque, J. Phys. I (France) **51**, 2515 (1990).
[19] R.A. Bagnold, *The Physics of Blown Sand and Desert Dunes* (Metheun, London, 1941); Proc. R. Soc. London A **295**, 219 (1966); R.L. Brown, J.C. Richard, *Principles of Power Mechanics* (Pergamon, New York, 1966).
[20] V.S. Lvov and I. Procaccia, Phys. Rev. Lett. **69**, 3543 (1992).
[21] G.A. Held, D.H. Solina, D.T. Keane, W.J. Haag, P.M. Horn, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).
[22] M. Bretz, J.B. Cunningham, P.L. Kurczynski, and F. Nori, Phys. Rev. Lett. **69**, 2431 (1992).
[23] H.M. Jaeger, C.-H. Liu, and S.R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).