Prediction of Turbulent Velocity Profile in Couette and Poiseuille Flows from First Principles

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To obtain the turbulent velocity profiles in Couette and Poiseuille flows between parallel walls, we use statistical mechanics of point vortices. Our theory does not contain phenomenological constants. The only parameter taken from experiments is the total kinetic energy of the flow. We solve numerically the equation for averaged flow and show that the theoretical predictions are in good agreement with the experimental data. [S0031-9007(96)00166-4]

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Characteristics of turbulent flow would be known if one knew the invariant measure of the attractor of viscous fluid motion. The complexity of this attractor does not leave a hope for “exact” knowledge of invariant measure, but some simple and effective approximations might be possible. Experiments show that fluctuations of the total energy of turbulent flows are small. This suggests that the attractor lies in close vicinity to the surface of constant energy in the phase space, and a natural approximation for the invariant measure seems to be the invariant measure of an ideal fluid flow, since its phase trajectories lie on energy surfaces. The situation is complicated, however, by the fact that the motion of an ideal fluid is not ergodic on the energy surface: Ideal fluid flow has an infinite number of additional integrals of motion, circulations of velocity over closed fluid circuits. Prescribing these integrals is equivalent to prescribing the initial vorticity. Each phase trajectory of ideal fluid flow belongs to a “sheet” in the phase space characterized by the value of energy and the initial vorticity field. We assume that the motion is ergodic on each sheet.

Ergodic motion on the sheets can be studied by means of approximation of vorticity by a large number N of vortex filaments. In a two-dimensional case, the theory of vortex gas is mature enough to predict average velocity profiles (see [1–7] and references therein), while determination of more delicate characteristics like velocity or pressure fluctuations is still an open problem.

The invariant measure of the attractor of viscous fluid motion can be approximated by the invariant measure of a sheet if a rule is established linking energy and initial vorticity of ideal fluid flow with parameters of turbulent flow. In this paper, we propose such a rule and argue that, in the limit of large Reynolds numbers, initial vorticity evolves to some “invariant” vorticity which is determined only by the total energy of the flow. The equation for invariant vorticity is established for channel flows. We compare the theoretical predictions of velocity profiles with the available experimental data and observe a remarkable coincidence.

The scheme of calculation of turbulent velocity profiles is as follows. Let us take, for simplicity, the Couette flow: The walls move in opposite directions with velocities u and $-u$. Consider values of u for which the flow is turbulent. Imagine a laminar flow with the same value of the wall velocity u. This flow is unstable with respect to finite disturbances. To find the evolution of the laminar flow after it loses its stability we take the vorticity field of this flow as initial vorticity for the flow of an ideal fluid. We introduce a grid of vortices dividing the flow field into a large number of pieces of equal area and substituting each piece by a point vortex with the corresponding intensity. Since vorticity of laminar Couette flow is constant, point vortices have equal intensities in this case. We then disturb the grid and study the evolution of the vortex system. To choose an appropriate disturbance we note that the energy of the flow is an integral of motion. Thus it should be taken equal to the energy of the turbulent flow under consideration. Therefore, we disturb the vortex positions in such a way that the energy of the disturbed vortex system has a prescribed value, and then let vortices go. Some turbulent flow is developed. Stream function of this flow is time independent if the number of vortices N tends to infinity, and the vortex motion is ergodic (see [1–7]). Stream function has fluctuations of order $1/\sqrt{N}$. Fluctuations of velocity may be finite. The final flow does not depend on the initial disturbance. In the limit $N \to \infty$, the “steady part” of the stream function ψ can be found from the equation [1,2]

$$-\Delta\psi(x) = \int_V \omega_0(\xi)\, \frac{e^{-\beta\omega_0(\xi)\psi(x)}}{\int_V e^{-\beta\omega_0(\xi)\psi(x')}dx'}\, d\xi,$$
$$\psi|_{\partial V} = 0\,, \qquad (1)$$

where ω_0 is the initial vorticity field, parameter β has the sense of inverse temperature of vortex motion, and Δ is Laplace operator. Parameter β is in one-to-one correspondence with the total kinetic energy of the flow E. Equation (1) is derived in [1] from the assumption that the motion of vortices is ergodic.

In the case of Couette flow ω_0 is constant and Eq. (1) is reduced to

$$-\Delta\psi = \omega_0\, \frac{e^{-\beta\omega_0\psi(y)}}{\frac{1}{|V|}\int_V e^{-\beta\omega_0\psi(y')}dy'}\,, \quad \psi|_{\partial V} = 0\,. \qquad (2)$$

Equation (2) can be studied analytically and numerically.

Equation (1) determines the operator of mixing, denoted by M, which maps an initial vorticity field ω_0 into a vorticity field of the final flow

$$\omega(x) = M\omega_0(x). \quad (3)$$

The explicit equation for the mixing operator allows one to discuss the following questions.

1. Let $\omega_0(x)$ be a vorticity field for which the flow loses its stability. Which vorticity field develops afterward? The transition from $\omega_0(x)$ to $\omega(x)$ shall further be referred to as primary mixing.

2. Change of flow parameters may cause the loss of stability for the flow that is formed by primary mixing. Then the next process of mixing occurs. Which vorticity field develops in this case? It shall further be referred to as secondary mixing. To find the flow after secondary mixing one has to plug into (1) instead of $\omega_0(x)$ the vorticity field that is formed by primary mixing.

3. Repeating the process of mixing many times we may arrive at the fixed point of mixing operator, the invariant flow ω_{inv},

$$\omega_{\text{inv}} = M\omega_{\text{inv}}. \quad (4)$$

It is natural to expect that flows with very high Reynolds numbers will obey Eq. (4) while flows with low Reynolds numbers are described by primary mixing and, therefore, should be governed by Eq. (3).

In the case of simply connected region V, the mixing operator is given by the left-hand side of Eq. (1). The mixing operator carries some specific features of the point vortex approximation. In particular, the maximum value of mixed vorticity field $\omega(x)$ may be larger than the maximum value of $\omega_0(x)$. This corresponds to the concentration of point vortices in some subregions. Note that mixing of a smooth flow of ideal fluid occurs in another way: Maximum vorticity after mixing is equal to maximum initial vorticity. Comparison with the experiments given below shows that the mixing operator based on the point vortex approximation seems to capture correctly the increase of vorticity in boundary layers.

The fixed points of the mixing operator are the solutions of the equation

$$\Delta\psi(x) = \int_V \Delta\psi(\xi)\,\frac{e^{\beta\Delta\psi(\xi)\psi(x)}}{\int_V e^{\beta\Delta\psi(\xi)\psi(x')}\,dx'}\,d\xi,$$
$$\psi|_{\partial V} = 0. \quad (5)$$

Equations (1) and (5) are highly nonlinear and highly nonlocal. Nevertheless, their numerical solutions can be obtained relatively easily.

For the flows between parallel walls region V can be taken as a long rectangle with periodic conditions at the short sides. In this case V is the double connected region. Additional analysis [2] shows that some modifications should be done in the expression for the mixing operator for such regions. First, for double connected regions, ψ should be taken constant at each of the two pieces of the boundary. Without the loss of generality, ψ can be taken zero at one piece of the boundary. Then the value ψ at the other piece of the boundary is equal to total discharge of the flow. Second, due to symmetry of the region an additional integral of motion appears,

$$\int_V \omega x_2\,dx = F. \quad (6)$$

Here x_2 is the coordinate which is normal to the wall. Existence of the additional integral leads to the appearance of an additional parameter α in Eq. (1):

$$-\Delta\psi(x) = \int_V \omega_0(\xi)\,\frac{e^{-\omega_0(\xi)[\beta\psi(x)+\alpha x_2]}}{\int_V e^{-\omega_0(\xi)[\beta\psi(x')+\alpha x_2']}dx'}\,d\xi. \quad (7)$$

Parameter α should be taken in such a way that the solution of Eq. (7) has a prescribed value of the integral (6). The above-mentioned modifications of the mixing operator for channel flows yield the corresponding modifications in the fixed point equation (5). Some comments are appropriate at this point.

1. If Eq. (5) is integrated over the region V, one gets an identity. Hence, one additional condition should be set in order to select a unique solution. We choose for this purpose the average vorticity of the flow,

$$-\frac{1}{V}\int_V \Delta\psi\,d^2x = \overline{\omega}. \quad (8)$$

Having prescribed average vorticity, we fix the value of average tangent velocity at the boundary.

2. Equation (5) does not admit pointwise no-slip boundary condition and only the average tangent velocity of the boundary can be given. If tangent velocity at the boundary is not constant along the boundary, molecular viscosity may contribute to the expression for the mixing operator. Fortunately, tangent velocity is constant along the boundary for the well documented Couette and Poiseuille flows, and we can check the theory in these cases.

3. Solutions of the fixed point equation for Couette flow follow the log-law asymptotically as Re $\to \infty$ ($E \to 0$) [2]. This stresses the point of view that molecular viscosity plays an important role only in formation of the boundary layer, but, being formed, the boundary layer is supported mostly by inertia forces.

Couette flow.—In the case of Couette flow we have to solve Eq. (7) in a rectangle with periodic boundary conditions on the short sides, and $\psi = 0$ on the long sides (the discharge of the flow is zero). All quantities in Eq. (7) are scaled by means of the wall velocity u and the distance h between the wall and the center of the channel. We keep the same notations for dimensionless quantities.

Denote the dimensionless coordinate which is orthogonal to the walls by y, $-1 \le y \le 1$. We are searching for solutions of Eq. (7) which depend only on y. Functions ψ and ω_0 are even functions of y. Therefore, integral (6) and constant α are zeros. Vorticity $\omega_0(y)$ for laminar flow is constant, $\omega_0 = -1$. Equation (7) takes the form

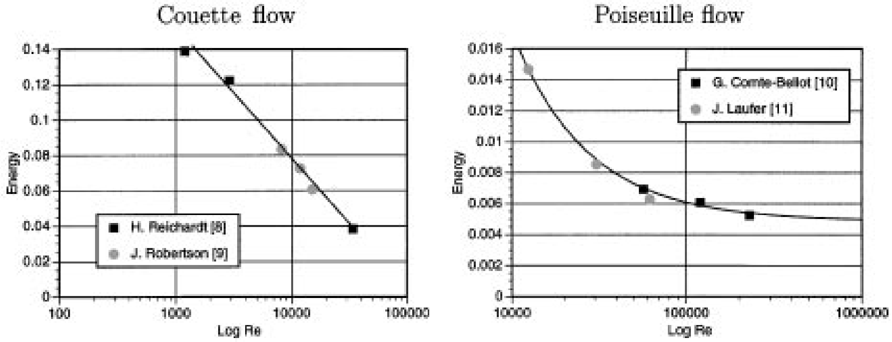

FIG. 1. Dependence of energy on Reynolds number for Couette and Poiseuille flows.

$$\psi_{yy} = \frac{2e^{\beta\psi(y)}}{\int_{-1}^{1} e^{\beta\psi(y')}\,dy'}, \quad \psi(-1) = \psi(1) = 0\,. \qquad (9)$$

Index y denotes the derivative with respect to y. Problem (9) has an analytical solution:

$$\psi(y) = \frac{2}{\beta}\ln\left(\frac{\cos(A)}{\cos(Ay)}\right). \qquad (10)$$

The constant A obeys the equation

$$\beta = 2A\tan A\,. \qquad (11)$$

Note that boundary conditions $\psi_y(\pm 1) = \pm 1$ are satisfied automatically because the mixing operator conserves the total vorticity.

Parameter β is determined by the prescribed value of kinetic energy E. From (10) and (11) we obtain the equation for β,

$$\frac{1}{2}\int_{-1}^{1}\psi_y^2\,dy = \frac{2}{\beta}\left(1 - \frac{2A^2}{\beta}\right) = E\,. \qquad (12)$$

Experimental dependence of E on the Reynolds number can be found from the experimental data by Reichardt [8] and Robertson [9], as shown in Fig. 1. Mention that the growth of Re is accompanied by the decay of energy in the coordinate system in which the discharge is equal to zero.

Velocity distribution for low Reynolds number Re = 2900 ($E = 0.122$ and $\beta = 11.097$) versus experimental data by Reichardt is presented in Fig. 2. As is seen, there is a good agreement between theory and experiment.

Agreement between the primary mixing profile and experimental data decays if Reynolds number grows. For high Reynolds numbers one has to use the equation of invariant mixing, which takes the form

$$\psi_{yy} = \int_{-1}^{1}\psi_{\xi\xi}(\xi)\,\frac{e^{\beta\psi_{\xi\xi}(\xi)\psi(y)}}{\int_{-1}^{1} e^{\beta\psi_{\xi\xi}(\xi)\psi(y')}\,dy'}\,d\xi\,,$$
$$\frac{1}{2}\int_{-1}^{1}\psi_y^2\,dy = E, \quad \psi(\pm 1) = 0, \quad \psi_y(\pm 1) = \pm 1\,. \qquad (13)$$

The latter boundary condition is a consequence of Eq. (8).

Equation (13) can be solved numerically. The solution for Re = 34 000 ($E = 0.03842, \beta = 20.5$) is shown in Fig. 2. It fits quite well the experimental data by Reichardt [8].

Poiseuille flow.—For Poiseuille flow the walls have equal velocities. It is convenient to choose a coordinate system in which the total discharge of the flow is zero. For definitness, we assume that average velocity at the center of the channel is positive. Then the wall velocities are negative and equal to $-u$. We scale all quantities by means of u and the distance h between the walls and the center of the channel. Dimensionless velocity is equal to -1 at the walls.

Dimensionless energy is a single-valued function of the Reynolds number. Energy dependence on the Reynolds number taken from experimental data by Comte-Bellot [10] and Laufer [11] is shown in Fig. 1. In Poiseuille flow, velocity ψ_y is an even function of y while ψ and ω_0 are odd functions. Since discharge is zero, $\psi(\pm 1) = 0$. In general, integral (6) and constant α are not zeros for Poiseuille flow. In the coordinate system chosen the integral (6) is equal to $2\psi_y(1)$. Thus, prescribing this integral is equivalent to prescribing ψ_y at the boundary. In accordance with the scaling chosen $\psi_y(\pm 1) = -1$. Condition (8) is satisfied automatically due to symmetry properties of the flow. So, to find the flow after primary mixing we have to solve the problem

$$\psi_{yy} = \int_0^1 \omega_0(\xi)\,\frac{\sinh\{\omega_0(\xi)[\beta\psi(y) + \alpha y]\}}{\int_0^1 \cosh\{\omega_0(\xi)[\beta\psi(y') + \alpha y']\}\,dy'}\,d\xi\,,$$
$$\psi(0) = 0, \quad \psi(1) = 0, \quad \psi_y(1) = -1, \quad \int_0^1 \psi_y^2 dy = E\,. \qquad (14)$$

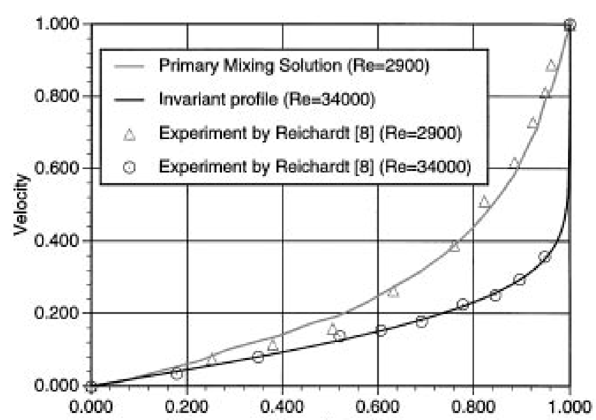

FIG. 2. Velocity distributions for Couette flow.

The last two conditions serve to determine the constants α and β. For laminar flow $\omega_0(y) = -3y$. Primary mixing for Re = 12 300 ($E = 0.014\,68$, $\alpha = -2.2775$, $\beta = 20.5026$) is shown in Fig. 3. It coincides quite well with the experimental data by Laufer [11]. For larger Reynolds numbers the agreement is not as close, but is still good. Considerable deviations are observed for Re > 50 000. We expected that for such high Reynolds numbers mixing should be invariant and the fixed point equation should take place. However, we faced the following difficulty. It turns out that the mixing operator does not have a smooth fixed point with a monotone velocity profile. To show that, let us write down the fixed point equation

$$\psi_{yy} = \int_0^1 \psi_{\xi\xi}(\xi) \frac{\sinh\{\psi_{\xi\xi}(\xi)[\beta\psi(y) + \alpha y]\}}{\int_0^1 \cosh\{\psi_{\xi\xi}(\xi)[\beta\psi(y') + \alpha y']\} dy} d\xi . \tag{15}$$

Integrating the absolute value of this equation over y and applying the integral inequality we have

$$\int_0^1 |\psi_{yy}(y)| dy \leq \int_0^1 |\psi_{\xi\xi}(\xi)| \frac{\int_0^1 |\sinh\{\psi_{\xi\xi}(\xi)[\beta\psi(y) + \alpha y]\}| dy}{\int_0^1 \cosh\{\psi_{\xi\xi}(\xi)[\beta\psi(y') + \alpha y']\} dy'} d\xi . \tag{16}$$

Since $\sinh x < \cosh x$, we arrive at the contradiction

$$\int_0^1 |\psi_{yy}(y)| dy < \int_0^1 |\psi_{\xi\xi}(\xi)| d\xi .$$

We had not been suspecting the absence of solutions when we started the numerical study of Eq. (15). In fact, some strange behavior of numerical solutions suggested questioning the existence of fixed points. However, numerical simulations revealed an interesting fact: There are quasisolutions of Eq. (15) that are functions which satisfy (15) with very high accuracy. The difference in the right-hand side and the left-hand side of (15) divided by the average value of the left-hand side may be of order 10^{-5}. It turns out that quasisolutions fit experimental data quite well. An example for Re = 120 000 is shown in Fig. 3. Note the coincidence of velocity profiles in the boundary layer. In other words, one can say that substitution of experimental data in Eq. (15) satisfies this equation with high accuracy. The numerical scheme used to solve Eq. (15) has been derived from the procedure of successive mixing. The velocity profile seems converging to Π-shaped profile. The experimental profiles are observed as intermediate steps in the course of iterations.

We assume that there should be a small term which is missing in Eq. (15). This term should restore the existence of solutions and should not affect quasisolutions. There are a number of reasons to improve Eq. (15). First, we did not take into account small corrections which are due to finiteness of the effective number of degrees of freedom in fluid motion (finiteness of the number of vortices in point vortex approximation). Second, we neglected three-dimensional effects. Third, we disregarded molecular viscosity. Which one of these reasons is crucial in restoring the existence of solutions is not clear at present.

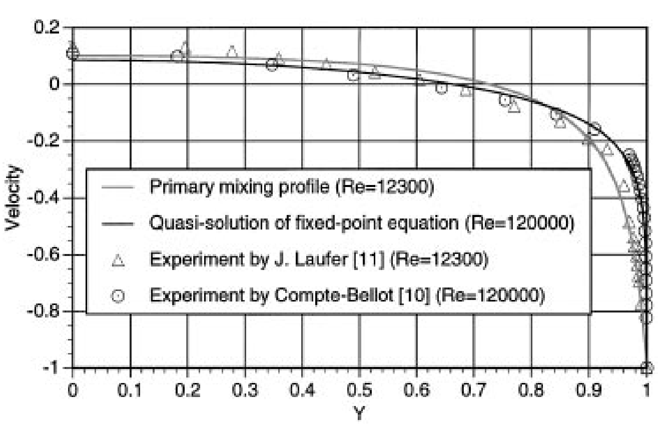

FIG. 3. Velocity profiles for Poiseuille flow.

The discussed examples demonstrate that statistical mechanics of point vortices satisfactorily predicts turbulent velocity profiles in channel flows. It is difficult to avoid the temptation to think that it happens not by chance, and that the mixing operator really captures some peculiarities of turbulent motion. However, many questions should be answered before this can be stated with full confidence.

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[1] V. L. Berdichevsky, Phys. Rev. E **51**, 4432 (1995).
[2] V. L. Berdichevsky, Phys. Rev. E (to be published).
[3] G. Joyce and D. Montgomery, J. Plasma Phys. **10**, 107–121 (1973).
[4] Y. B. Pointin and T. S. Lundgren, Phys. Fluids **19**, 1459–1470 (1976).
[5] J. Miller, Phys. Rev. Lett. **65**, 2137–2140 (1990).
[6] R. Robert, J. Stat. Phys. **65**, 531–551 (1991).
[7] A. Shnirelman, Russ. J. Math. Phys. **1**, 105–114 (1993).
[8] H. Reichardt, Max-Planck-Institute Technical Report No. 22, 1959.
[9] J. M. Robertson, in 5th Midwestern Conference on Fluid Dynamics (1957).
[10] G. Comte-Bellot, *Ecoulement Turbulent Entre deux parois paralleles* (Ministere DF L'Air, Paris, France, 1965).
[11] J. Laufer, J. Aeronaut. Sci. **17**, 277–287 (1950).