Optical Solitons in Presence of Kerr Dispersion and Self-Frequency Shift

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We consider the higher order nonlinear Schrödinger (HNLS) equation describing nonlinear wave propagation in light guides with all higher order effects such as higher order dispersion, Kerr dispersion, and stimulated inelastic scattering. Using the Painlevé analysis, we derive all parametric conditions for soliton-type pulse propagation in HNLS fiber system. We generalize the Ablowitz-Kaup-Newell-Segur method to the 3×3 eigenvalue problem, and constructed the Lax pair for the integrable case. The one soliton solution is generated from a Bäcklund transformation and an exact N-soliton solution is explicitly obtained from the Hirota bilinearization. The significance of the soliton solution is discussed. [S0031-9007(96)00071-3]

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Intelligence transmission using optical fibers is the technology of choice for the next generation of optical fiber communication. The recent progress of research on all-optical soliton transmission systems have revealed that they can overcome the limitations on the speed and distance of linear wave transmission systems. Looking into the growth of the technology behind optical solitons, it took nearly seven years for an experimental observation of optical solitons [1], since its first theoretical report by Hasegawa and Tappert [2]. The reason for this is because the amount of constraints in conducting an experiment on optical solitons are enormous.

Optical soliton in fiber is possible because of the exact balancing between the group velocity dispersion (GVD) and its counterpart self-phase modulation (SPM) [2,3]. SPM is the nonlinear effect due to the lowest dominant nonlinear susceptibility $\chi^{(3)}$ in silica fibers [3]. Most of the nonlinear effects due to $\chi^{(3)}$ will not be in fibers as they need a phase matching condition to be satisfied. But for ultrashort pulses, in addition to the SPM, $\chi^{(3)}$ will produce higher order nonlinear effects like the self-steepening (otherwise called the Kerr dispersion) and the stimulated Raman scattering (SRS). Apart from GVD, the ultrashort pulse will also suffer from third order dispersion (TOD).

Wave propagation in optical fibers with these effects is governed by the higher order nonlinear Schrödinger (HNLS) equation. Kodama *et al.* [4–6] derived this HNLS equation, and using perturbation theory they treated all higher order terms as perturbation to the nonlinear Schrödinger (NLS) soliton. In 1986 Mitschke and Mollenauer [7] reported the self-frequency shift of the NLS soliton due to Raman effect. For large channel handling capacity and for high speed it is necessary to transmit solitons at a high bit rate of ultrashort pulses. So it is very important that all higher order effects be considered in the propagation of femtosecond pulses. For optical pulses higher order effects such as TOD, Kerr dispersion, and SRS give asymmetrical broadening either temporally or spectrally. In contrast GVD and SPM produce symmetric broadening in the time and frequency domain, respectively, and counterbalance for a parametric condition (anomalous dispersion regime) to propagate bright solitons. Similarly there can be some possibilities to have soliton propagation with all higher order effects which induce asymmetrical broadening. In recent years, many authors have analyzed the HNLS equation from different points of view [5–7,9]. To our knowledge nobody has reported the Lax pair, N-soliton solutions, and other related properties of solitons in the HNLS system.

In this Letter, using the Painlevé analysis, we derive the parametric conditions for soliton-type pulse propagation in the HNLS fiber system. To construct the linear eigenvalue problem for the integrable case, we generalize the 2×2 Ablowitz-Kaup-Newell-Segur (AKNS) [8] method to the 3×3 eigenvalue problem, and the Lax pair is constructed. We also generate the one soliton solution from a Bäcklund transformation and, for the first time, we obtain explicitly exact N-soliton solutions from the Hirota bilinearization. Finally, we discuss the significance of the soliton solution.

Kodama and Hasegawa [4] in 1985 derived the HNLS equation which describes wave propagation in a nonlinear fiber medium with higher order effects such as TOD, Kerr dispersion, and SRS. Normally the dispersion due to TOD will be negligible when compared to GVD. But a considerable amount of asymmetrical broadening in the time domain will be produced by TOD for ultrashort pulses. The self-steepening, otherwise called the Kerr dispersion, is due to the intensity dependence of group velocity. This forces the peak of the pulse to travel slower than the wings, which causes an asymmetrical spectral broadening of the pulse. SRS gives a self-frequency shift to the pulse. The self-frequency shift is a self-induced redshift in the pulse spectrum arising from SRS. The long wavelength components of the pulse experience Raman gain at the expense of the short wavelength components,

resulting in an increasing redshift as the pulse propagates. It has been recognized that the self-frequency shift is a potentially detrimental effect in soliton communication systems because power fluctuations at the source translate into frequency fluctuations in the fiber through the power dependence of the soliton self-frequency shift and hence into timing jitter at the receiver [10]. With all these effects, the governing dimensionless envelope wave equations (HNLS) takes the form

$$E_z = i(\alpha_1 E_{tt} + \alpha_2|E|^2E) + \varepsilon[\alpha_3 E_{ttt} + \alpha_4(|E|^2E)_t + \alpha_5 E(|E|^2)_t], \quad (1)$$

where E is the slowly varying envelope of the electric field, the subscripts z and t are the spatial and temporal partial derivatives, and α_1, α_2, α_3, α_4, and α_5 are the parameters related to GVD, SPM, TOD, self-steepening, and SRS, respectively.

Equation (1) for $\varepsilon = 0$ reduces to the NLS equation [2,3]. The NLS equation includes only the GVD and SPM effects well known in the fiber, and it allows soliton-type wave propagation in the anomalous dispersion regime (bright soliton useful for optical communication). For $\alpha_3 = \alpha_5 = 0$, Eq. (1) describes the derivative NLS (DNLS) equation. DNLS governs the propagation of NLS soliton in the presence of Kerr dispersion. The Kerr dispersion is seldom treated as a perturbation to soliton propagation. But the DNLS system also allows soliton-type pulse propagation. So, the HNLS system in general does not admit soliton-type pulse propagation but in the limiting cases admits several soliton possessing systems. In order to identify the conditions for soliton-type pulse propagation, we apply the Painlevé analysis [11–16].

The parametric conditions for which any NPDE allows soliton-type pulse propagation can be systematically derived using the Painlevé analysis. For the Painlevé analysis, we introduce a new set of variables a $(=E)$ and b $(=E^*)$. Using this in (1), a and b can be written as

$$\begin{aligned} a_z &= i(\alpha_1 a_{tt} + \alpha_2 a^2 b) + \varepsilon[\alpha_3 a_{ttt} + \alpha_4(a^2b)_t + \alpha_5 a(ab)_t], \\ b_z &= -i(\alpha_1 b_{tt} + \alpha_2 b^2 a) + \varepsilon[\alpha_3 b_{ttt} + \alpha_4(b^2a)_t + \alpha_5 b(ba)_t]. \end{aligned} \quad (2)$$

The generalized Laurent series expansions of a and b are

$$\begin{aligned} a &= \varphi^{\mu} \sum_{r=0}^{\infty} a_r(z,t)\varphi^r, \\ b &= \varphi^{\delta} \sum_{r=0}^{\infty} b_r(z,t)\varphi^r, \end{aligned} \quad (3)$$

with $a_0, b_0 \neq 0$, where μ and δ are negative integers, and a_r and b_r are the set of expansion coefficients which are analytic in the neighborhood of the noncharacteristic singular manifold $\varphi(z,t)$. Looking at the leading order, we substitute $a \approx a_0\varphi^{\mu}$ and $b \approx b_0\varphi^{\delta}$ into Eq. (2), and upon balancing the dominant terms we obtain the following results:

$$\mu = \delta = -1 \quad \text{and} \quad a_0 b_0 = \frac{-6\alpha_3}{3\alpha_4 + 2\alpha_5}\varphi_t^2. \quad (4)$$

Substituting the full expansion of the Laurent series and keeping only the leading order terms, we obtain the following resonances:

$$r = -1, 0, 3, 4, 3 \pm 2\sqrt{\frac{-\alpha_5}{3\alpha_4 + 2\alpha_5}}. \quad (5)$$

The resonance $r = -1$ corresponds to the arbitrariness of the singularity manifold and $r = 0$ corresponds to the fact that either a_0 or b_0 is arbitrary. Upon substituting the full Laurent series into Eq. (2) and collecting the coefficients of the different powers of φ we find that Eq. (2) admits a sufficient number of arbitrary functions only for the conditions

$$\alpha_1 = \tfrac{1}{2}, \alpha_2 = 2 \quad \text{and } \alpha_3{:}\alpha_4{:}(\alpha_4 + \alpha_5) = 1{:}6{:}3, \quad (6a)$$

$$\alpha_1 = 1, \alpha_2 = 4 \quad \text{and } \alpha_3{:}\alpha_4{:}(\alpha_4 + \alpha_5) = 1{:}6{:}3. \quad (6b)$$

Hence we can say that the HNLS equation allows soliton-type pulse propagation only for these parametric restrictions. With the conditions (6a) and (6b) the HNLS equation takes the following form:

$$E_z = i\left(\tfrac{1}{2}E_{tt} + 2|E|^2E\right) + \varepsilon[E_{ttt} + 6|E|^2E_t + 3E(|E|^2)_t], \quad (7a)$$

$$E_z = i(E_{tt} + 4|E|^2E) + \varepsilon[E_{ttt} + 6|E|^2E_t + 3E(|E|^2)_t]. \quad (7b)$$

In Ref. [17] Sasa and Satsuma have shown that Eq. (7a) can be transformed into a complex modified Korteweg–de Vries (KdV) equation (with the SPM parameter = 1). Using the suitable transformations, they have transformed the HNLS equation to the complex modified KdV equation and then solved the complex modified KdV equation for the soliton solution using IST. The shape of the one soliton they have reported is very peculiar (singular) with two peaks. From the optical soliton communication point of view it is very difficult to generate such a kind of soliton pulse shape using soliton lasers. Here we derive a simple sech shape for the HNLS soliton. Before constructing the soliton solutions, first we derive the Lax pair of Eq. (7a).

We generalize the 2×2 AKNS method to the 3×3 eigenvalue problem, and we derive the Lax pair for the HNLS equation (7a) in the form

$$\Psi_t = U\Psi, \qquad \Psi = (\Psi_1\Psi_2\Psi_3)^T, \qquad \Psi_z = V\Psi, \quad (8)$$

where

$$U = \begin{pmatrix} -i\lambda & E & E^* \\ -E^* & i\lambda & 0 \\ -E & 0 & i\lambda \end{pmatrix}, \quad (9a)$$

$$V = (2i\lambda^2 - 8i\varepsilon\lambda^3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & E & E^* \\ -E^* & 0 & 0 \\ -E & 0 & 0 \end{pmatrix}(\lambda - 4\varepsilon\lambda^2)$$
$$+ \left(\frac{i}{2} - 2i\varepsilon\lambda\right)\begin{pmatrix} 2|E|^2 & E_t & E_t^* \\ E_t^* & -|E|^2 & -(E^*)^2 \\ E_t & -E^2 & -|E|^2 \end{pmatrix} + \varepsilon\begin{pmatrix} 0 & E_{tt} + 4|E|^2E & E_{tt}^* + 4|E|^2E^* \\ -(E_{tt}^* + 4|E|^2E^*) & -(E_tE^* - EE_t^*) & 0 \\ -(E_{tt} + 4|E|^2E) & 0 & -(E_t^*E - E^*E_t) \end{pmatrix}, \tag{9b}$$

where λ is the isospectral parameter. Using the compatibility condition $U_z - V_t + [U, V] = 0$, one can derive the HNLS equation (7a).

To obtain the soliton solution from the Lax pair, we use the Darboux-Bäcklund transformation method [18,19]. The one soliton solution of Eq. (7a) is obtained in the form

$$E = \sqrt{2}\,\beta\,\mathrm{sech}[2\beta t + 8\varepsilon\beta^3 z]\exp(2i\beta^2 z), \tag{10}$$

where $\lambda = i\beta$ and the ratio of integration constants is assumed to be equal to $\sqrt{2}$.

Because of the complicated structure of the Lax pair it is tedious to generate multisoliton solutions. So, we obtain the exact N-soliton solution for the HNLS equation using the Hirota direct method [20]. For this we use the dependent variable transformations [20,21]

$$E(z, t) = \frac{G(z, t)}{F(z, t)}, \tag{11}$$

where $G(z, t)$ is a complex function and $F(z, t)$ is a real function with respect to z and t. Using Eq. (11), Eq. (1) may be decoupled into bilinear equations, with the condition $\alpha_1(3\alpha_4 + 2\alpha_5) = 3\alpha_2\alpha_3$, in the form

$$(iD_z + \alpha_1 D_t^2 - i\varepsilon\alpha_3 D_t^3)G \cdot F = 0, \tag{12a}$$
$$\alpha_1 D_t^2 F \cdot F = \alpha_2 GG^*, \tag{12b}$$
$$D_t G \cdot G^* = 0. \tag{12c}$$

In Eq. (12) we use the Hirota D operator defined by

$$D_z^m D_t^n (G \cdot F) = \left[\frac{\partial}{\partial z} - \frac{\partial}{\partial z'}\right]^m \left[\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right]^n \times G(z, t)F(z', t')\Big|_{z'=z, t'=t}. \tag{13}$$

However, from the results of Painlevé analysis, the HNLS equation is integrable only for conditions (6a) and (6b). Hence substituting conditions (6a) and (6b) into Eq. (12) the bilinear form for Eqs. (7a) and (7b) can be obtained.

The exact solutions of the bilinear form of Eq. (7a) can be expressed as

$$F(z, t) = \sum_{\beta=0,1} \exp\left[\sum_{i,j(i<j)}^{(2N)} \rho_{ij}\beta_i\beta_j + \sum_{i=1}^{2N} \beta_i\zeta_i\right], \tag{14a}$$

$$G(z, t) = \sum_{\gamma=0,1} \exp\left[\sum_{i,j(i<j)}^{(2N)} \rho_{ij}\gamma_i\gamma_j + \sum_{i=1}^{2N} \gamma_i\zeta_i\right], \tag{14b}$$

with

$$\zeta_j = X_j z + \eta_j t + \zeta_j^0, \quad X_j = \frac{i\eta_j^2}{2} + \varepsilon\eta_j^3,$$
$$\zeta_{j+N} = \zeta_j^*, \quad X_{j+N} = X_j^*, \quad \eta_{j+N} = \eta_j \quad \text{for } j = 1, 2, \ldots, N, \tag{15}$$

$$\rho_y = \begin{cases} \ln\frac{2}{(\eta_i + \eta_j)^2} & \text{for } i = 1, 2, \ldots, N \quad \text{and } j = N+1, N+2, \ldots, 2N, \\ \ln\frac{(\eta_i - \eta_j)^2}{2} & \text{for } i = 1, 2, \ldots, N \quad \text{and } j = 1, 2, \ldots, N, \\ & \text{or } i = N+1, N+2, \ldots, 2N \quad \text{and } j = N+1, N+2, \ldots, 2N, \end{cases} \tag{16}$$

where η_j is a real parameter, ζ_j^0 is a complex constant; $\sum_{\beta=0,1}$ indicates the summation over all the possible combinations of $\beta_1 = 0, 1, \beta_2 = 0, 1, \ldots, \beta_{2N} = 0, 1$ under the condition $\sum_{i=1}^N \beta_i = \sum_{i=1}^N \beta_{i+N}$; $\sum_{\gamma=0,1}$ indicates the summation over all the possible combinations of $\gamma_1 = 0, 1, \gamma_2 = 0, 1, \ldots, \gamma_{2N} = 0, 1$ under the condition $\sum_{i=1}^N \gamma_i = \sum_{i=1}^N \gamma_{i+N}$; and $\sum_{i,j>i=1}^{(2N)}$ indicates the summation over all the possible pairs taken from $2N$ elements with the specified condition $j > 1$, as indicated. We assume all η_i are different from each other.

From Eqs. (14)–(16), we generate the one soliton solution of Eq. (7a) in the form

$$E(z, t) = \frac{\eta}{\sqrt{2}}\,\mathrm{sech}[\eta(t + \varepsilon\eta^2 z)]\exp\left(\frac{i\eta^2 z}{2}\right). \tag{17}$$

From Eqs. (10) and (17) it is clear that the one soliton solution obtained for the HNLS equation (7a) is the same (with $\eta = 2\beta$).

For completeness, we also give the two soliton solution of Eq. (7a) which reads

$$E = \frac{(\sqrt{2}\,\eta^+/|\eta^-|)\{\eta_1 \cosh[\eta_2 t + \varepsilon\eta_2^3 z]\exp(i\eta_1^2 z/2) + \eta_2 \cosh[\eta_1 t + \varepsilon\eta_1^3 z]\exp(i\eta_2^2 z/2)\}}{\cosh[\eta^+ t + \varepsilon(\eta_1^3 + \eta_2^3)z] + (\eta^+/\eta^-)^2 \cosh[\eta^- t + \varepsilon(\eta_1^3 - \eta_2^3)z] + [4\eta_1\eta_2/(\eta^-)^2]\cos[(\eta^+\eta^-/2)z]}, \tag{18}$$

where $\eta^+ = \eta_1 + \eta_2$ and $\eta^- = \eta_1 - \eta_2$.

From the (integrability) condition $\alpha_1(3\alpha_4 + 2\alpha_5) = 3\alpha_2\alpha_3$ (or $4\alpha_1 = \alpha_2$), it is clear that one can also construct the N-soliton solutions for the above parametric condition and Eq. (7b) can be related to Eq. (7a) under certain transformation in E.

Thus the one soliton shape of the HNLS fiber system is a simple sech shaped one, unlike the results of Sasa and Satsuma [17]. So the simple sech shaped initial soliton pulse can be easily produced from a soliton laser.

In Eqs. (10) and (17) if we take the limit $\varepsilon \to 0$, the solution becomes the NLS equation soliton solution [3]. It is interesting to see that if we put $\alpha_5 = 0$ in Eq. (1), the system reduces to the extended NLS equation explained by Liu and Wang [21], and the corresponding soliton solution is also found to be the same.

To conclude, for the first time, from the Painlevé analysis we derived two parametric conditions between the parameters of GVD, SPM, TOD, Kerr dispersion, and SRS, for which the HNLS fiber system allows soliton-type pulse propagation. Thus the asymmetrical broadening between the three higher order effects balance themselves, giving rise to soliton pulse propagation. We have generalized the 3×3 linear eigenvalue problem for the AKNS method and systematically derived the Lax pair for the HNLS equation. The exact N-soliton solution for the HNLS fiber system is explicitly generated using the Hirota direct method, and the one soliton solution is compared with that of the soliton solution obtained from Bäcklund transformations. The single soliton is a simple sech shape (unlike the peculiar shape given in [17]), so it can be easily produced in a soliton laser. Hence we believe that all the results we have presented in this Letter will be very useful for the reality of all soliton communication links.

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[1] L. F. Mollenauer, R. H. Stolen, and J. P. Gordon, Phys. Rev. Lett. **45**, 1095 (1980).
[2] A. Hasegawa and F. Tappert, Appl. Phys. Lett. **23**, 142 (1973).
[3] G. P. Agrawal, *Nonlinear Fiber Optics* (Academic, San Diego, 1989).
[4] Y. Kodama and A. Hasegawa, IEEE J. Quantum Electron. **QE-23**, 510 (1987).
[5] Y. Kodama, J. Stat. Phys. **39**, 597 (1985).
[6] A. Hasegawa, *Optical Solitons in Fibers* (Springer, Heidelberg, 1989); A. Hasegawa and Y. Kodama, *Solitons in Optical Communications* (Oxford University Press, Oxford, 1996).
[7] F. M. Mitschke and L. F. Mollenauer, Opt. Lett. **11**, 657 (1986).
[8] M. J. Ablowitz, D. J. Kaup, A. C. Newell, and H. Segur, Phys. Rev. Lett. **31**, 125 (1973).
[9] K. Porsezian, P. Shanmugha Sundaram, and A. Mahalingham, Phys. Rev. E **50**, 1543 (1994); K. Nakkeeran and K. Porsezian, Opt. Commun. **123**, 169 (1996).
[10] D Wood, IEEE J. Lightwave Tech. **8**, 1097 (1990).
[11] J. Weiss, M. Tabor, and G. Carnevale, J. Math. Phys. (N.Y.) **24**, 522 (1983).
[12] M. D. Kruskal (private communication).
[13] M. Musette and R. Conte, J. Math. Phys. (N.Y.) **32**, 1450 (1991).
[14] K. Porsezian and M. Lakshmanan, J. Math. Phys. (N.Y.) **32**, 2932 (1991).
[15] K. Porsezian and K. Nakkeeran, Phys. Rev. Lett. **74**, 2941 (1995).
[16] K. Nakkeeran and K. Porsezian, J. Phys. A **28**, 3817 (1995).
[17] N. Sasa and J. Satsuma, J. Phys. Soc. Jpn. **60**, 409 (1991).
[18] K. Porsezian, J. Phys. A **24**, L337 (1991).
[19] K. Porsezian and K. Nakkeeran, J. Mod. Opt. **42**, 1953 (1995).
[20] R. Hirota, J. Math. Phys. (N.Y.) **14**, 805 (1973).
[21] S. Liu and W. Z. Wang, Phys. Rev. E **49**, 5726 (1994).