Comment on "Universal Scaling Functions in Critical Phenomena"

Recently, Hu, Lin, and Chen (HLC) [1] studied the university of the finite size scaling functions (FSSF) by simulating percolation on several planar lattices with free (f) and periodic (p) boundary conditions (bc). Simulating the probability $R_L(p)$ that a system of size L percolates at site (or bond) occupancy p, near the percolation threshold p_c, they presented their data (collected for a single value of L) in the scaling form $R_L(p) = F(D(p - p_c)L^{1/\nu}) \equiv F(Dx) \equiv F(\hat{x})$, with the *nonuniversal* metric scale factor D having the same value for all problems on a given lattice [including different bc and also the percolation probability $P_L(p) = L^{-\beta/\nu}S(\hat{x})/D_3$], and with the *universal* FSSF $F(\hat{x})$ [and $S(\hat{x})$] depending only on the bc. HLC claimed that $R_L(p)$ is not related to the "free energy," and, therefore, the above results extend the earlier published scope [2]. The fact that D (and D_3) was the same for all problems was also presented as surprising.

In this Comment we show that the "surprising" numerical results of HLC are direct consequences of the renormalization group (RG), as discussed in this context in Ref. [3]. Furthermore, HLC's numerical data for the small-$\hat{x}$ expansion of F for fbc are in agreement with those published in Ref. [3]. However, their results for pbc strongly disagree with those recently obtained by us [4].

The RG maps any initial problem onto another problem, which is close to the RG fixed point (fp). This new problem is described by the linearized scaling fields near the fp, including the "temperaturelike" scaling field, μ, the symmetry breaking "ghost" field at site j, H_j, and the irrelevant fields, ω_i. Given these fields, the singular part of the free energy density (i.e., the statistical generating function), f, is completely determined by the neighborhood of the fp, and is therefore universal. Using the linearized recursion relations for the various fields, we have $f = b^{-d}f(\mu b^{1/\nu}, H_j b^{y_h}, \omega_i b^{-\theta_i}, L/b)$, with the length rescale factor b [3]. The only nonuniversal aspect here involves the mapping of the initial parameters p and ghost fields h_j onto the variables which appear in f. Since $\mu = 0$ when $p = p_c$, one expects to leading order $\mu = D(p - p_c)$. Similarly, $H_j = Bh_j$. Note that the scale factors D and B depend only on the transient stage of mapping our problem to the vicinity of the fp, and have nothing to do with the specific quantity which is being measured. In contrast to HLC, we now argue that *both* $R_L(p)$ and $P_L(p)$ are obtainable as derivatives of f with respect to some h_j at $h_j = 0$. For example, for fbc, $R_L(p)$ is the conditional probability that one of the sites on one edge of the sample is connected to another site at the opposite edge. This can clearly be expressed in terms of correlation functions, which are derivatives of f with respect to the ghost fields on these two edges. Therefore $R_L(p)$ is also a function of the *same* scaled variables as f above, with the same scale factors D and B (which determines HLC's D_3). The same will apply to pbc, and certainly to the simpler case of $P_L(p)$.

To obtain the FSSF, one sets $b = L$. This results with corrections to scaling terms. For example, the coefficients of the small-$\hat{x}$ expansion of F depend on L via $\omega_i L^{-\theta}$, allowing for an efficient (and cheap) extrapolation to $L \to \infty$ [3,4]. These corrections cannot be seen from a single L simulation, as used by HLC. In contrast, we studied $R_L(p)$ for several L's. For example, in [3] we measured F for fbc, with $L \le 40$ [typically 40 different p's and $(2-5) \times 10^6$ samples per p] finding that $F(\hat{x}) = \frac{1}{2} + \hat{x} + K_3\hat{x}^3 + \cdots$, with the universal amplitude ratio $K_3 = -1.02 \pm 0.02$ (the metric factor D is normalized by setting $\frac{\partial F}{\partial \hat{x}}|_{\hat{x}=0} = 1$). For the site problem on the square lattice we measured $D_{\text{site,sq}} = 0.760 \pm 0.005$. These results are in agreement with, and even more accurate than, HLC's data which give $K_3 \approx -1.1$, $D_{\text{site,sq}} \approx 0.762$.

More recently [4], we studied pbc by simulating the site and bond problems on the square lattice at p_c. Generalizing the hull generating walk method [5], we collected 10^6–10^7 samples with sizes up to $L = 1000$. Extrapolating to $L \to \infty$, we find $R_\infty(p_c) = F(0) = 0.63\,665 \pm 0.0008$ [4]. This agrees well with the recent Monte Carlo data for helical bc, $R_\infty(p_c) \approx 0.63$ [6], which are in this case very similar to the pbc, but strongly disagrees with HLC's $F(0) = 0.93(4)$. Additional simulations, using $L \le 64$, and collecting $(1-2) \times 10^6$ samples for 60 different p's, yielded $F(\hat{x}) = F(0) + \hat{x} + K_2\hat{x}^2 + K_3\hat{x}^3 + K_4\hat{x}^4 + \cdots$, with the universal ratios $K_{2,3,4} = -0.517 \pm 0.010, -1.08 \pm 0.1, 0.94 \pm 0.15$. This is again very different from HLC's $K_{2,3,4} \approx -5.9, 11.8, 30.3$. We suspect HLC used some unconventional pbc.

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