Poliatzky Replies: In the case of a Dirac equation with a central potential $\lambda V(r)$ one has to differentiate between the weak and strong forms of Levinson's theorem. The weak form is a statement about the sum of positive and negative energy phase shifts (see Ref. [1] for more details):

$$\eta_{m\kappa}(0)+\eta_{-m\kappa}(0) = \begin{cases} (N_\kappa^+ + N_\kappa^-)\pi, & |\kappa| = 1,2,\dots, \\ (N_\kappa^+ + N_\kappa^- + \frac{1}{2})\pi, & |\kappa| = 1, \end{cases} \quad (1)$$

where $\eta_{\pm m\kappa}(0)$ are the phase shifts at threshold energies $\epsilon = \pm m$, N_κ^+ is the number of positive, and N_κ^- is the number of negative energy bound states. The first case refers to a situation without a threshold resonance (a finite and yet not normalizable solution at threshold energies $\epsilon = \pm m$) and the second case refers to a situation with a threshold resonance. As explained in Ref. [1] the main physical message of the weak form of Levinson's theorem (as well as of Levinson's theorem for the Schrödinger equation) is the completeness of states. Consequently, it is a rather general statement and is valid even in the case of a strong potential with $|\lambda V(r)| > 2m$ in some region of the radial variable r.

The strong form of Levinson's theorem is a statement about the positive and negative energy phase shifts separately and it is also related to the completeness of states, but the situation is more subtle here. In Ref. [2] it was shown that

$$\eta_{\pm m\kappa}(0) = \begin{cases} n_\kappa^\pm \pi, & |\kappa| = 1,2,\dots, \\ (n_\kappa^\pm + \frac{1}{2})\pi, & \kappa = \mp 1, \end{cases} \quad (2)$$

where $n_\kappa^\pm$ are the numbers of nodes of the large component of the Dirac wave function at energies $\epsilon = \pm m$, respectively (in notations of Ref. [2] the large component is $u_{1\epsilon\kappa}$ at positive and $u_{2\epsilon\kappa}$ at negative energies). An equivalent statement is to say that n_κ^+ and n_κ^- are the numbers of bound states which in the process of increasing the coupling constant from 0 to λ have entered the spectral gap at $\epsilon = m$ and $\epsilon = -m$, respectively. Levinson's theorem as given by (2) is valid only for the case of a weak potential: $|\lambda V(r)| < 2m$ for all r. This has been correctly pointed out by Ma [3]. In the case of a strong potential a new phenomenon occurs which reflects the possibility of a pair creation in the vicinity of such a potential. Below we give a brief description of this phenomenon and a generalization of (2) to the case of a strong potential.

The way (2) was proved is to convert the Dirac equation to a pair of effective Schrödinger equations [see (22) and (23) of Ref. [2]] and then to use the usual connection between the phase shift at zero energy and the number of bound states (equivalently the number of nodes). This proof fails in the case of a strong potential because at the point r where $|\lambda V(r)| = 2m$ the effective potential becomes singular in a way that a simultaneous vanishing of the radial wave function and its first derivative becomes possible. Consequently, at this critical point r pairs of nodes can be created and annihilated as a result of changing the coupling constant and hence the usual connection between nodes, bound states, and phase shifts is no longer valid. However, the fact that the phase shift increases by π (decreases by $-\pi$), if a node enters (leaves) the interval $(0,\infty]$ through $r = \infty$, remains valid. Hence, as a result of changing the coupling constant from 0 to λ the phase shift accumulates a value which is determined by the difference between the total number of nodes which have entered and the total number of nodes which have left the interval $(0,\infty]$ through $r = \infty$. Therefore, generally Levinson's theorem states that

$$\eta_{\pm m\kappa}(0) = \begin{cases} (n_{<\kappa}^\pm - n_{>\kappa}^\pm)\pi, & |\kappa| = 1,2,\dots, \\ (n_{<\kappa}^\pm + \frac{1}{2} - n_{>\kappa}^\pm)\pi, & \kappa = \mp 1, \\ (n_{<\kappa}^\pm - n_{>\kappa}^\pm - \frac{1}{2})\pi, & \kappa = \mp 1, \end{cases} \quad (3)$$

where $n_{<\kappa}^\pm$ ($n_{>\kappa}^\pm$) is the number of nodes of the large component of the Dirac wave function at positive $(+)$ and negative $(-)$ energies, which in the process of increasing the coupling constant from 0 to λ has entered $(<)$ [left $(>)$] the interval $(0,\infty]$ through $r = \infty$, and $+\frac{1}{2}\pi$ ($-\frac{1}{2}\pi$) is for the case of a threshold resonance with a node at $r = \infty$ which is entering (leaving) $(0,\infty]$. An equivalent statement is to say that $n_{<\kappa}^\pm$ ($n_{>\kappa}^\pm$) is the total number of bound states which have entered (left) the spectral gap at threshold energies $\epsilon = \pm m$, respectively. The statement (3) of the strong form of Levinson's theorem, while equivalent to (2) in the case of a weak potential where everywhere $|\lambda V(r)| < 2m$, remains valid even in the case of a strong potential where in some region $|\lambda V(r)| > 2m$. Notice that the weak form (1) follows from the strong form (3).

The formal proof of the above statements proceeds using a strategy similar to the one of Ref. [4] which deals with a similar problem. More details are given in Ref. [5].

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[1] N. Poliatzky, Helv. Phys. Acta **66**, 241 (1993). See also *ibid.* **67**, 683 (1994).
[2] N. Poliatzky, Phys. Rev. Lett. **70**, 2507 (1993).
[3] Z.-Q. Ma, preceding Comment, Phys. Rev. Lett. **76**, 3654 (1996).
[4] A. Martin and J. Stubbe, J. Math. Phys. **36**, 4680 (1995).
[5] N. Poliatzky, Report No. ETH-TH/96-06.

