Comment on "Levinson's Theorem for the Dirac Equation"

Levinson's theorem [1] is one of the fundamental theorems in quantum scattering theory. For the Schrödinger equation with a nonsingular spherically symmetric potential, it gives a quantitative relation between the limit of the phase shift at zero energy and the number of bound states. The latter is equal to the number of nodes of the radial function at zero energy due to the Sturm-Liouville theorem.

Barthélémy [2] first discussed Levinson's theorem for the Dirac equation by the generalized Jost function. He stated that Levinson's theorem was valid for positive and negative energies separately as in the nonrelativistic case. But later this statement was found incorrect for a strong potential [3,4]. The correct statement of Levinson's theorem for the Dirac equation is [3,4]

$$N_\kappa = \{\delta_\kappa(M) + \delta_\kappa(-M)\}/\pi - \{\sin^2[\delta_\kappa(M)] + \sin^2[\delta_\kappa(-M)]\}/2\,, \quad (1)$$

where κ is the standard angular momentum parameter. N_κ and $\delta_\kappa(E)$ are the number of bound states and the phase shift at energy E for the given angular momentum κ, respectively. The last term in (1) stands for the half bound state, which was first shown by Newton [5]. The main character of Levinson's theorem for the Dirac equation is that the limits of the phase shifts at the thresholds $E = \pm M$ (zero momentum) may be negative, in comparison with the phase shifts of free particle. For example, as an attractive potential becomes strong enough, a scattering state of positive energy may change to a bound state and, furthermore, change to a scattering state of negative energy. In this case the limit $\delta_\kappa(-M)$ of phase shift at $E = -M$ becomes negative.

Recently, Poliatzky [6,7] transformed the Dirac equation into a couple of effective Schrödinger-type equations near the thresholds $E = \pm M$ and showed a stronger statement of Levinson's theorem for the Dirac equation:

$$n_\kappa(\pm M) = \delta_\kappa(\pm M)/\pi - \sin^2[\delta_\kappa(\pm M)]/2\,, \quad (2)$$

where $n_\kappa(\pm M)$ denotes the numbers of nodes of certain radial functions with energies $\pm M$, respectively [8]. In Ref. [6] $n_\kappa(\pm M)$ were explained as the numbers of bound states of certain effective Schrödinger-type equations, that were not easy to count. For the potential $|V(r)| < 2M$, two numbers are equal to each other due to the Sturm-Liouville theorem.

It is easy to see that the stronger statement (2) of Levinson's theorem for the Dirac equation is incorrect for a strong potential ($|V| > 2M$), because the limit of the phase shifts at thresholds for the Dirac equation may be negative, but the number of nodes, or the number of bound states, is a non-negative integer. Why may the limit of the phase shift be negative for an effective Schrödinger-type equation? The key point is that the effective potential in the effective Schrödinger-type equations is singular when the potential V is finite, but strong enough [see Eqs. (22) and (23) in [6] when $|V| = \pm 2M$]. For those singular equations Levinson's theorem does not hold. For the same reason, Poliatzky's proof for the statement that the sum of the number of nodes, $n_\kappa(M) + n_\kappa(-M)$, is equal to the number of bound states N_κ is also wrong for the strong potential. Therefore, Poliatzky proved his stronger version of Levinson's theorem for the Dirac equation only for a weak potential ($|V| < 2M$).

It can also be explicitly seen from a calculable example. As an example consider the Dirac equation with a square well potential: $V(r) = -\lambda$ when $r \le r_0$, and $V(r) = 0$ when $r > r_0$. In the notation of Ref. [4] there are two radial functions $f_{\kappa E}(r)$ and $g_{\kappa E}(r)$ that are proportional to u_1 and $-u_2$ used in [6], respectively. In the following we discuss only the solution with $\kappa > 0$. The solution with $\kappa < 0$ is similar [4].

It is straightforward to solve the radial functions at the energies $E = \pm M$ [4]. Carefully counting the nodes of $f_{\kappa \pm M}(r)$ and $g_{\kappa \pm M}(r)$ when λ increases and decreases from zero, one can find that if $\delta_\kappa(M) \ge 0$, $\delta_\kappa(M)/\pi$ is equal to the number of nodes of $g_{\kappa M}(r)$, which is the same as that of $f_{\kappa M}(r)$; if $\delta_\kappa(M) < 0$, $-\delta_\kappa(M)/\pi$ is equal to the number of nodes of $g_{\kappa M}(r)$ in the region $0 < r < r_0$; if $\delta_\kappa(-M) \ge 0$, $\delta_\kappa(-M)/\pi$ is equal to the number of nodes of $g_{\kappa -M}(r)$; and if $\delta_\kappa(-M) < 0$, $-\delta_\kappa(-M)/\pi$ is equal to the number of nodes of $g_{\kappa -M}(r)$ in the region $0 < r < r_0$ subtracting the number of its nodes in the region $r > r_0$. Therefore, the stronger statement (2) is correct for a weak potential ($|V| < 2M$), but incorrect for a strong potential ($|V| > 2M$) where the phase shift at $E = M$ or $E = -M$ becomes negative.

Zhong-Qi Ma
Institute of High Energy Physics
P.O. Box 918(4)
Beijing 100039, People's Republic of China

Received 30 May 1995 [S0031-9007(96)00122-6]
PACS numbers: 03.65.Ge

[1] N. Levinson and K. Danske, Vidensk. Selsk. Mat-fys. Medd. **25** (1949).
[2] M.-C. Barthélémy, Ann. Inst. Henri Poincaré **A7**, 115 (1967).
[3] Zhong-Qi Ma, Phys. Rev. D **32**, 2203 (1985).
[4] Zhong-Qi Ma and Guang-Jiong Ni, Phys. Rev. D **31**, 1482 (1985).
[5] R. G. Newton, J. Math. Phys. (N.Y.) **1**, 319 (1960).
[6] N. Poliatzky, Phys. Rev. Lett. **70**, 2507 (1993).
[7] N. Poliatzky, Helv. Phys. Acta **66**, 241 (1993).
[8] N. Poliatzky, Report No. hep-th/9411197.

 0031-9007/96/76(19)/3654(1)$10.00