Electromagnetic Radiation by a Tunneling Charge

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Electromagnetic radiation during the quasiclassical tunneling motion of a charge through a smooth potential barrier is considered. A general formula for the radiation spectral density per tunneling particle is derived, which is essentially classical (i.e., does not contain the Planck constant), and is given by a simple modification of the convenient result for the classical overbarrier motion. [S0031-9007(96)00117-2]

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A charge moving in an external potential experiences acceleration and hence emits electromagnetic radiation. For the simple case of a one-dimensional infinite motion the classical formula describing the bremsstrahlung is [1]

$$\frac{\partial \mathcal{E}}{\partial \omega} = \frac{2}{3\pi} \frac{e^2}{c^3} |a_\omega|^2, \tag{1}$$

where $\partial \mathcal{E}/\partial \omega$ is the total radiation energy per unit frequency interval, e is the particle charge, c is the velocity of light, and a_ω is the Fourier transform of the charge acceleration $a(t)$,

$$a_\omega = \int_{-\infty}^{\infty} a(t) \exp(-i\omega t)\, dt\,. \tag{2}$$

In quantum mechanics, by applying the Fermi golden rule one obtains the same result if (i) the charge motion is quasiclassical and (ii) the emitted photon energy, $\hbar\omega$, is much less than the particle energy E.

Quantum mechanics also says that in the presence of a potential barrier some particles may be transmitted through the barrier for $E < U_0$, where U_0 is the maximum barrier height. In the quasiclassical situation the fraction of these particles is exponentially small, so their contribution to the total emitted radiation is negligible. However, one may be specifically interested in the bremsstrahlung during this underbarrier motion, and this is the problem that we address in this Letter [2].

Assuming that conditions (i) and (ii) are fulfilled, we show that in the case of quantum tunneling the bremsstrahlung spectrum is described by the same classical equation (1), in which, however, in order to get a_ω one should take the integral in Eq. (2) along a certain contour in the complex plane of variable t. We also show that for the underbarrier motion the value a_ω in Eq. (1) may be found as an analytical continuation of the classical function $a_\omega(E)$ from region $E > U_0$ to the region $E < U_0$.

We stress that the situation is rather unusual, since the expression for the radiation spectral density per tunneling particle is classical: it does not contain the Planck constant $\hbar$, though the number of such particles, of course, goes to zero in the classical limit.

Our starting point is the well-known quantum formula for the bremsstrahlung spectrum [3]. For the one-dimensional nonrelativistic case considered here it may be written as

$$\left(\frac{\partial \mathcal{E}}{\partial \omega}\right)_f = \frac{2}{3\pi} \frac{e^2}{c^3} |a_{if}|^2 \frac{1}{W_f}, \tag{3}$$

where $(\partial \mathcal{E}/\partial \omega)_f$ is the radiation energy in a unit frequency interval per particle in the final state f, W_f is the probability of a transition to this state, a_{if} is the matrix element of the acceleration operator $a(z) = -m^{-1}\partial U(z)/\partial z$ between the initial and final states, m is the particle mass, and $U(z)$ is the potential energy. The corresponding wave functions are normalized to unit flux.

For a given ω there are two final states, $f = r, t$ where r stands for reflection and t for transmission. Accordingly, $W_t = T$ and $W_r = R$, where T and R are the transmission and reflection coefficients, respectively. The total spectrum is given by

$$\frac{\partial \mathcal{E}}{\partial \omega} = T\left(\frac{\partial \mathcal{E}}{\partial \omega}\right)_t + R\left(\frac{\partial \mathcal{E}}{\partial \omega}\right)_r. \tag{4}$$

If conditions (i) and (ii) are satisfied, for $E < U_0$ we have $R \approx 1$ and $T \ll 1$, so the value $\partial \mathcal{E}/\partial \omega \approx (\partial \mathcal{E}/\partial \omega)_r$ is given by the classical equation (1). We are interested in finding $(\partial \mathcal{E}/\partial \omega)_t$ which gives the radiation spectrum produced by a single tunneling particle.

For the initial state we take the wave function $\Psi_k^{(+)}$ describing an incoming particle moving in the positive direction of the z axis [Fig. 1(a)]. According to the Sommerfeld rule for the calculation of bremsstrahlung [3] the final state should be described by the function $\Psi_{-k}^{(-)}$ for the reflected particles, and by the function $\Psi_k^{(-)}$ for the transmitted ones [Fig. 1(b)]. The functions $\Psi_{\pm k}^{(+)}$ and $\Psi_{\pm k}^{(-)}$ are related by

$$\Psi_k^{(-)} = (\Psi_{-k}^{(+)})^*, \qquad \Psi_k^{(+)} = A\Psi_{-k}^{(-)} + B\Psi_k^{(-)},$$

where A and B are the amplitudes of the reflected and transmitted waves, respectively (see Fig. 1), so that $R = |A|^2$, $T = |B|^2$.

We consider the radiation during transmission, therefore

$$a_{if} = \int_{-\infty}^{\infty} (\Psi_{k_i}^{(+)})^* a(z) \Psi_{k_f}^{(-)}\, dz\,. \tag{5}$$

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Above the barrier the quasiclassical functions $\Psi_{k_i}^{(+)}$ and $\Psi_{k_f}^{(-)}$ may be written in the form

$$\Psi_{k_i}^{(+)} = \left(\frac{m}{p_i(z)}\right)^{1/2} \exp\left[ik_i(z - z_0) + i\int_{-\infty}^{z}\left(\frac{1}{\hbar}\, p_i(z_1) - k_i\right) dz_1\right], \tag{6}$$

$$\Psi_{k_f}^{(-)} = \left(\frac{m}{p_f(z)}\right)^{1/2} \exp\left[ik_f(z - z_0) - i\int_{z}^{\infty}\left(\frac{1}{\hbar}\, p_f(z_1) - k_f\right) dz_1\right., \tag{7}$$

where $\hbar k_i = (2mE)^{1/2}$, $p_i(z) = \{2m[E - U(z)]\}^{1/2}$, the quantities k_f, $p_f(z)$ are defined analogously, with the initial energy E replaced by the final energy $E_f = E - \hbar\omega$. The phase of the function $\Psi_{k_i}^{(+)}$ is chosen in such a way that for $z \to -\infty$ we have $\Psi_{k_i}^{(+)} \propto \exp[ik_i(z - z_0)]$, where z_0 is an arbitrary point. The function $\Psi_{k_f}^{(-)}$ has a similar asymptotic form at $z \to \infty$.

For $E < U_0$ the quasiclassical expressions for $\Psi_{k_i}^{(+)}$ and $\Psi_{k_f}^{(-)}$ differ from those given by Eqs. (6) and (7) only by additional reflected waves existing at $z < a$ for $\Psi_{k_i}^{(+)}$ and at $z > b$ for $\Psi_{k_f}^{(-)}$. Here a and b are the left and right turning points, respectively. However, one can see that the relative contribution of the reflected waves to the matrix element a_{if} is exponentially small, being proportional to an integral from a rapidly oscillating function. Hence for calculating a_{if} one may use the functions given by Eqs. (6) and (7) both for $E > U_0$ and for $E < U_0$. In the latter case at $a < z < b$ the function $p_i(z)$ should be understood as $i|p_i(z)|$, while $p_f(z) = -i|p_f(z)|$. Such a choice of the branches of $p(z)$ provides the correct behavior of the two functions within the underbarrier region.

A straightforward calculation gives

$$a_{if} = \exp[i\Phi + i\omega\Delta t(z_0)] \times \int_{-\infty}^{\infty} \frac{dz}{v(z)}\, a(z) \exp\left(-i\omega \int_{z_0}^{z} \frac{dz_1}{v(z_1)}\right), \tag{8}$$

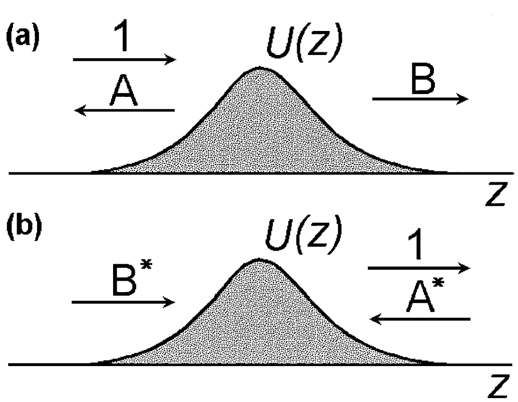

FIG. 1. Schematic representation of the initial (a) and final (b) states for calculating the matrix element in Eq. (5). (a) Function $\Psi_k^{(+)}$ describing the incoming particles; (b) function $\Psi_k^{(-)}$ describing the transmitted particles. The 1, A, and B above the arrows denote the amplitudes of corresponding waves.

where

$$\Phi = \int_{-\infty}^{\infty} dz\left(k_i - \frac{1}{\hbar}\, p_i^*(z)\right), \tag{9}$$

$$\Delta t(z_0) = \int_{z_0}^{\infty} dz\left(\frac{1}{v(z)} - \frac{1}{v_0}\right), \tag{10}$$

$v(z) = \{2[E - U(z)]/m\}^{1/2}$, $v_0 = v(\pm\infty) = (2E/m)^{1/2}$. In deriving Eqs. (8)–(10) we used the relation $p_i^*(z) - p_f(z) = \hbar\omega/v(z)$.

Equations (8)–(10) are valid both for $E > U_0$ and $E < U_0$. Moreover, the matrix element a_{if} is an analytical function of energy E. For $E > U_0$ the quantities Φ and $\Delta t(z_0)$ are real. Φ is the total phase difference introduced by the potential $U(z)$, and $\Delta t(z_0)$ has the physical meaning of the time delay with respect to free motion from point z_0 to ∞.

For $E < U_0$ at $a < z < b$ the function $v(z)$ in Eqs. (8) and (10) should be understood as $-i|v(z)|$. Thus Φ becomes complex. In fact, $B = \exp(i\Phi)$ is the complex amplitude of the transmitted wave and $T = |B|^2$ is the exponentially small (for $E < U_0$) transmission coefficient. The time delay $\Delta t(z_0)$ generally becomes complex too. It remains real if the point z_0 is chosen to the right of the turning point b.

We now rewrite Eq. (8) by introducing instead of z a new time variable

$$t = \int_{z_0}^{z} \frac{dz_1}{v(z_1)}. \tag{11}$$

Then

$$a_{if} = B\exp[i\omega\Delta t(z_0)]a_\omega , \tag{12}$$

$$a_\omega = \int_C a(t)\exp(-i\omega t)\, dt , \tag{13}$$

where the integration contour C is defined by the manner in which the time t changes, according to Eq. (11), when z runs from $-\infty$ to ∞ along the real axis.

Note that a_ω in Eq. (13) depends on the choice of the point z_0, at which $t(z_0) = 0$ [this is true also for the classical equation (2)]. If we take another real point, z_0', in place of z_0 and choose $t(z_0') = 0$, the quantity a_ω changes by an additional factor

$$\exp\left(i\omega \int_{z_0}^{z_0'} \frac{dz}{v(z)}\right).$$

For $E > U_0$ this is just a phase factor. However, if $E < U_0$ and if the interval (z_0, z_0') happens to include at least a part of the underbarrier region, the modulus of this factor is no longer equal to unity. This change in the absolute value of a_ω is compensated by the corresponding change in the modulus of the factor $\exp[i\omega\Delta t(z_0)]$ in Eq. (12). The product

$$\tilde{a}_\omega = a_\omega \exp[i\omega\Delta t(z_0)] \qquad (14)$$

would always change by just a phase factor $\exp[i\omega(z_0' - z_0)/v_0]$, both for $E < U_0$ and for $E > U_0$. Thus $\tilde{a}_\omega$ depends on z_0 only through an irrelevant phase factor $\exp(i\omega z_0/v_0)$ for any energy.

Equation (12) gives the general correspondence between the matrix element a_{if} and the Fourier component a_ω for the quasiclassical infinite motion. As we have seen, $|a_{if}|^2 = |a_\omega|^2$ for $E > U_0$; however, this is no longer true for $E < U_0$. Comparing the classical equation (1) with the quantum equation (3) and using Eq. (12) we see that for arbitrary energy the quantity a_ω in Eq. (1) should be replaced by $\tilde{a}_\omega$ given by Eqs. (13) and (14).

The quantity $\tilde{a}_\omega$ is an analytical function of energy E. Thus we have the following rule for calculating the radiation spectrum of a tunneling charge. Given a classical a_ω value for $E > U_0$ and some choice of the point z_0, it should be multiplied by the factor $\exp[i\omega\Delta t(z_0)]$ to obtain $\tilde{a}_\omega$, which should then be analytically continued to the region $E < U_0$. The resulting value of $\tilde{a}_\omega$ should be put in Eq. (1) instead of a_ω. If z_0 is chosen to the right of the barrier top, then for energies $E > U(z_0)$ the delay $\Delta t(z_0)$ is real and the quantities $\tilde{a}_\omega$ and a_ω differ by a phase factor only. Thus for this special choice of z_0 and for the energy range $U(z_0) < E < U_0$ Eq. (1) is valid, with a_ω given by Eq. (13). However, in the general case one should replace a_ω by $\tilde{a}_\omega$.

We will now consider the integration contour in Eq. (13). At $E > U_0$ it obviously goes along the real axis (dashed line in Fig. 2). At $E < U_0$ the contour C is presented by the heavy line in Fig. 2, where for convenience we have chosen $t = 0$ at $z = b$. It runs from $-\infty - i\tau$ to $-i\tau$ (classical motion to the left of the turning point a), from $-i\tau$ to 0 (tunneling motion between turning points a and b), and from 0 to ∞ (classical motion at $z > b$). The quantity τ is the "bounce" time for tunneling [4]

$$\tau = \int_a^b \frac{dz}{|v(z)|}\,.$$

The function $a(t)$ is an analytical function of t in the complex t plane with cuts which start at the branching points. The branching points are situated at $t = t(z_i)$ where z_i are the poles of $U(z)$. The function $a(t)$ is periodic along the Imt axis with a period 2τ, since the underbarrier motion in imaginary time is equivalent to a periodic classical motion in an inverted potential. Note that the values of $a(t)$ along C are real. Obviously the contour C may be deformed at one's convenience without crossing the cuts. We note that a similar integration contour was introduced by Ivlev and Mel'nikov [5] who considered tunneling through a barrier in an external ac field.

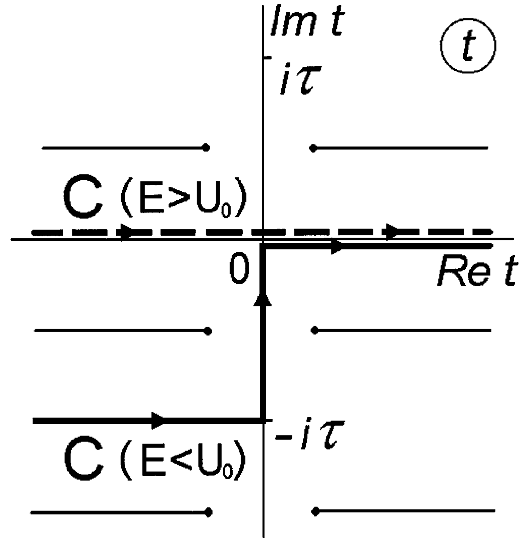

FIG. 2. Complex t plane with cuts and integration contours in Eq. (13) for $E > U_0$ and for $E < U_0$.

We will now derive asymptotic formulas for the radiation spectrum of a tunneling charge in the low-frequency and high-frequency limits. In the first case ($\omega\tau \ll 1$) we rewrite Eq. (13) as

$$a_\omega = i\omega \int_C [v(t) - v_0] \exp(-i\omega t)\, dt\,. \qquad (15)$$

Using Eqs. (10), (11), and (15) for $\omega \to 0$ we obtain $a_\omega = -i\omega v_0 \Delta t$, where $\Delta t = \Delta t(-\infty)$ is the total time delay given by Eq. (10). Finally, the radiation spectral density is

$$\left(\frac{\partial \mathcal{E}}{\partial \omega}\right)_t = \frac{2}{3\pi}\frac{e^2}{c^3}\,\omega^2 v_0^2 |\Delta t|^2. \qquad (16)$$

This formula is valid for both the classical motion above the barrier and the tunneling motion. The time delay, Δt, is an analytical function of energy which may be continued from the classical region $E > U_0$ to the tunneling region $E < U_0$. The ω^2 dependence at low frequencies is characteristic for the one-dimensional motion.

In the high-frequency case ($\omega\tau \gg 1$) it is convenient to connect the branching points in Fig. 2 by vertical cuts and shift the integration contour C far downwards so that it gets caught on cuts. The integral along the contour C would be equal to the sum of integrals around these cuts; however, for $\omega\tau \gg 1$ the contribution of the nearest to the real axis cut dominates. Calculations give the following result in the high-frequency limit: $(\partial\mathcal{E}/\partial\omega)_t \propto \omega^{2n/(n+2)} \exp(-\omega\tau)$, where n is the order of the pole of $U(z)$ which is nearest to the Rez axis.

We note that the relative contribution of the part of the contour C running from $-\infty - i\tau$ to $-i\tau$ to the integral in Eq. (13) is proportional to $\exp(-\omega\tau)$ and hence suppressed. This is related to the fact that if the particle emits a photon before tunneling it loses the energy $\hbar\omega$, and therefore its transmission coefficient decreases. Thus at $\omega\tau \gg 1$ there is only a small fraction of transmitted particles which have radiated a photon before entering the barrier.

As an example, we give explicit asymptotic results obtained for the potential $U(z) = U_0/\cosh^2(z/d)$,

$$\left(\frac{\partial \mathcal{E}}{\partial \omega}\right)_t = \frac{2e^2}{3\pi c^3} v_0^2 f(\omega\tau), \tag{17}$$

$$f(\xi) = \begin{cases} [\frac{1}{\pi^2}\ln^2(\frac{E}{U_0 - E}) + 1]\xi^2 & (\xi \ll 1), \\ 2(U_0/E)^{1/2}\xi\exp(-\xi) & (\xi \gg 1), \end{cases} \tag{18}$$

where $\tau = \pi d/v_0$.

In summary, we have considered electromagnetic radiation accompanying the quasiclassical tunneling motion of a charge through a potential barrier. We have shown that the radiation spectral density per tunneling particle is given by a classical formula (not containing the Planck constant), which is a simple modification of the conventional result for the classical overbarrier motion. We have also derived general asymptotic expressions for the radiation spectrum in the low- and high-frequency limits.

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[2] There may be situations (e.g., α decay) when the radiation is produced only by particles, which are transmitted through the barrier. In such cases the motion in the underbarrier region gives an appreciable contribution to the total emitted radiation.

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