Condition of Correspondence between Quantum and Classical Dynamics for a Chaotic System

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The paper deals with the dynamics of a semiclassical system under the influence of the environment. The effect of the environment is shown to convert the quantum dynamics of the system into the classical one. The condition of correspondence between quantum and classical dynamics is obtained and checked numerically.

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The transition from quantum to classical dynamics is a problem which has caused permanent interest since the foundation of quantum mechanics. One of the motivations for this interest comes from the well known experimental fact that the same quantum system can behave in different laboratory conditions either as a quantum system or a classical one. It seems that common agreement on the explanation of such a phenomenon has been achieved. The key idea is the following: There are no systems which are completely isolated from their environment, and this influence can be important. In fact, it has been numerically shown for some model systems that the influence of the environment does convert the quantum dynamics of the system into the classical one [1–5]. The aim of the present paper is to obtain a general condition under which the transition (due to the environment influence) from quantum to classical dynamics is achieved; here we are focused on chaotic systems.

The starting point of our analysis is the semiclassical formula for propagation of a quantum particle, known as the Van Vleck–Gutzwiller formula [6]

$$G(X,Y,t) = \sum_\alpha g_\alpha(X,Y,t) \equiv \sum_\alpha \left(-\frac{1}{2\pi i\hbar}\frac{\partial^2 S^\alpha(X,Y,t)}{\partial X\partial Y}\right)^{1/2} \times \exp\left[\frac{i}{\hbar}S^\alpha(X,Y,t) + \frac{i\pi}{2}\nu_\alpha\right]. \quad (1)$$

In Eq. (1) $S_\alpha(X,Y,t)$ is the action along the classical path connecting two points X and Y, ν_α is the Maslov index, and α labels different classical trajectories connecting X and Y. (For the sake of simplicity we consider the one-dimensional case.) One might doubt if the semiclassical approximation can be grounds for studying the transition between quantum and classical dynamics for a chaotic system. In fact, it is known that in the case of chaotic dynamics the time of correspondence between pure quantum and classical evolution is extremely small and scales as $t_c \sim \ln(1/\hbar)$ [7]. Fortunately, this does not imply that the semiclassical approximation will fail after t_c. It was shown in the papers of Heller and Tomsovic [8] that formula (1) describes the quantum dynamics of a classically chaotic system surprisingly well for time much larger than t_c.

The second key point of the analysis given below is the way in which the influence of the environment is taken into account. The common way is to construct a master equation for the system density matrix and then to solve it. Recently great progress has been achieved in solving the master equation by using the stochastic Schrödinger equation. The main idea is the following. Having solved the master equation, we can put it into correspondence with the Schrödinger-like equation (linear or nonlinear) which contains an additional stochastic term [9]. Instead of solving the master equation, we solve the stochastic equation for a large number of the realizations of the stochastic process $\xi = \xi(t)$. Then the solution for the master equation is the average over ξ from pure density matrix

$$\rho(x',x'',t) = [\psi(x',t)\psi^*(x'',t)]_\xi\ . \quad (2)$$

In what follows we use the simplest form of the stochastic Schrödinger equation

$$i\hbar\frac{\partial\psi(t)}{\partial t} = \hat{H} + \epsilon\xi(t)\hat{V}\psi(t),$$
$$\overline{\xi(t)\xi(t')} = \delta[(t-t')/\tau]. \quad (3)$$

It is easy to show that Eq. (3) corresponds to the following equation for the system density matrix:

$$\frac{\partial\hat\rho(t)}{\partial t} = -\frac{i}{\hbar}[H,\hat\rho(t)] - \frac{\epsilon^2\tau}{\hbar^2}\hat{V},[\hat{V},\hat\rho(t)], \quad (4)$$

which has a typical structure of the master equation for an open system, and where $\hat{V}$ is the operator of the system interaction with the environment and τ has physical sense of the correlation time for the environment variables.

We now proceed to the analysis. Let us denote the Wigner function of the system under consideration by

$$w(X,P,t) = \frac{1}{2\pi\hbar}\int dx\, \exp\left(\frac{iPx}{\hbar}\right) \times \rho\left(X-\frac{x}{2}, X+\frac{x}{2}, t\right). \quad (5)$$

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In Eq. (5) $\rho(x', x'', t)$ is given by Eq. (2). Since $\psi(x,t) = \int dy\, G(x,y,t)\psi(y,0)$ we obtain after a simple transformation the following expression for the semiclassical evolution of the Wigner function:

$$w(X,P,t) = \int\int\left[\sum_{\alpha,\beta} f_{\alpha,\beta}(X,P;Y,Q;t)\right] w(Y,Q,0)\,dY\,dQ\,, \tag{6}$$

where

$$f_{\alpha\beta}(X,P;Y,Q;t) = \left\{\frac{1}{2\pi\hbar}\int\int \exp\left(i\frac{Px}{\hbar}\right) g_\alpha\left(X-\frac{x}{2}, Y-\frac{y}{2}, t\right) g_\beta^*\left(X+\frac{x}{2}, Y+\frac{y}{2}, t\right)\exp\left(-i\frac{Qy}{\hbar}\right)dx\,dy\right\}_\xi \tag{7}$$

and $g_\alpha(X,Y,t)$ is given by Eq. (1).

Now we shall show that the terms with $\alpha = \beta$ in Eq. (6) define pure classical evolution of the system. In fact, substituting $g_\alpha(X \mp x/2, Y \mp y/2, t)$ into Eq. (7) results in the following form:

$$g_\alpha\left(X \mp \frac{x}{2}, Y \mp \frac{y}{2}, t\right) \approx \left(\frac{-1}{2\pi i\hbar}\frac{\partial^2 S^\alpha}{\partial X\partial Y}\right)^{1/2} \times \exp\left(\frac{i}{\hbar}\left[S^\alpha \mp \frac{\partial S^\alpha}{\partial X}\frac{x}{2} \mp \frac{\partial S^\alpha}{\partial Y}\frac{y}{2} + \frac{\partial^2 S^\alpha}{\partial X^2}\frac{x^2}{8} + \frac{\partial^2 S^\alpha}{\partial Y^2}\frac{y^2}{8} + \frac{\partial^2 S^\alpha}{\partial X\partial Y}\frac{xy}{4}\right] + \frac{i\pi}{2}\nu_\alpha\right) \tag{8}$$

[here $S^\alpha \equiv S^\alpha(X,Y,t)$]. We obtain $\sum_\alpha f_{\alpha,\alpha} \times (X,P;Y,Q;t) = \sum_\alpha \{-(\partial P_t^\alpha(X,Y)/\partial Y)\delta[P - P_t^\alpha \times (X,Y)]\delta[Q - P_0^\alpha(X,Y)]\}_\xi$, where $P_t^\alpha(X,Y) = \partial S^\alpha(X,Y,t)/\partial X$ and $P_0^\alpha(X,Y) = -\partial S^\alpha(X,Y,t)/\partial Y$ are the initial and final momentum of a classical particle moving from Y to X. Then, using the identity $\delta[\chi(\phi) - \chi_0]d\chi/d\phi = \delta[\phi - \chi^{-1}(\chi_0)]$, we come to the final expression [10]

$$\sum_\alpha f_{\alpha,\alpha}(X,P;Y,Q;t) = \{\delta[Y - X_0(X,P,t)] \times \delta[Q - P_0(X,P,t)]\}_\xi\,. \tag{9}$$

In Eq. (9) $X_0 = X_0(X,P,t)$, $P_0 = P_0(X,P,t)$ is the solution of the classical equation of the motion back in time and we omit the sum over α while for given X and P there is only one classical trajectory with the specified initial coordinate Y and momentum Q.

Having the result of the previous paragraph in mind we conclude that the transition to classical dynamics takes place under the condition of vanishing of the terms with $\alpha \neq \beta$ in Eq. (6). Let us obtain this condition. The use of the stationary phase method for $\alpha \neq \beta$ brings the complex prefactor $\exp\{(i/\hbar)[S^\alpha(X,Y,t) - S^\beta(X,Y,t)]\}$. Therefore, we can estimate $f_{\alpha,\beta}(X,P;Y,Q;t)$ in the order of magnitude as $f_{\alpha,\beta}(X,P;Y,Q;t) \sim \langle\exp\{(i/\hbar)[S^\alpha(X,Y,t) - S^\beta(X,Y,t)]\}\rangle_\xi$. Let us denote by $S_0(X,Y,t)$ the value of the principal Hamilton function for $\epsilon = 0$ and by $\delta S(X,Y,t)$ the variation of $S(X,Y,t)$ due to the stochastic term $[S(X,Y,t) = S_0(X,Y,t) + \delta S(X,Y,t)]$ and let $x_0(t)$ be the trajectory connecting the points X, Y in the case $\epsilon = 0$ $[x(t) = x_0(t) + \delta x(t)]$. We restrict ourselves in the case when the operator $\hat{V}$ of the system interaction with the environment is a function of the coordinate x and let the characteristic length for the variation of $V(x)$ coincide with the characteristic length for the variation of the potential energy $U(x)$. In the first order over parameter ϵ we have $S(X,Y,t) = \int_0^t \times [m\dot{x}^2/2 - U(x) - \epsilon\xi(5)V(x)]dt \approx S_0(X,Y,t) + \epsilon\int_0^t \times [m\dot{x}_0\delta\dot{x} - U'(x_0)\delta x]dt - \epsilon\int_0^t \xi(t)V(x_0)dt = S_0(X,Y,t) - \epsilon\int_0^t \xi(t)V(x_0)dt$ [we have used $\delta x(0) = \delta x(t) = 0$ and $m\ddot{x}_0 = -U'(x_0)$]. Thus the estimate for $f_{\alpha,\beta}(X,P;Y,Q;t)$ is the following:

$$f_{\alpha,\beta}(X,P;Y,Q;t) \sim \left\{\exp\left[\frac{i\epsilon}{\hbar}\int_0^t \xi(t)[V(x_0^\alpha) - V(x_0^\beta)]dt\right]\right\}_\xi \sim \left\{\exp\left[\frac{i\epsilon V}{\hbar}\int_0^t \xi(t)dt\right]\right\}_\xi = \exp(-\kappa t),$$
$$\kappa = \frac{\epsilon^2 V^2 \tau}{2\hbar^2}\,. \tag{10}$$

In Eq. (10) we denote by V the amplitude of $V(x)$, and use the fact that the typical distance between two different chaotic trajectories x_0^α and x_0^β coincides with the characteristic length for variation of the potential $U(x)$ and, hence, $V(x)$.

Now we are in a position to formulate the condition of the correspondence. We consider the case of the chaotic dynamics. In this case the number of classical trajectories contributing to Eq. (1) grows exponentially with an increment η defined by the Lyapunov exponent

λ: $\eta = \eta(\lambda) \sim \lambda$. Then the overall contribution of the terms with $\alpha \neq \beta$ can be neglected if and only if $2\eta < \kappa$. Thus the condition required has the form

$$\frac{\epsilon^2 V^2 \tau}{\hbar^2} > \eta(\lambda) . \qquad (11)$$

We note that a similar condition was obtained in Ref. [4] by means of a qualitative analysis of the master equation (4) in Wigner representation. The method presented in this paper is more rigorous and, in principle, allows us to predict an exact border between the quantum and classical dynamics.

To demonstrate the applicability of the condition (11) we perform a numerical simulation for a particular chaotic system known as the standard map on the torus. The Hamiltonian of this system has the form

$$H = \frac{p^2}{2} + U(x) \sum_n \delta(t - n) ,$$
$$U(x + 2\pi) = U(x) , \qquad (12)$$

where the periodic boundary condition on p is imposed. In the quantum case the condition of periodicity over the momentum is satisfied by the choice of the Planck constant $\hbar = 2\pi/N$. Then the Hamiltonian (12) defines an N-level system with basis functions $|n\rangle = (2\pi)^{-1/2} \exp(inx_k)$, $x_k = 2\pi k/N$.

Since the phase space of the system (12) is bounded ($0 \leq x < 2\pi$, $0 \leq p < 2\pi$), the Wigner function is defined on the grid and transformation (5) takes the form [11]

$$w(k, l, t) = \sum_{j=0}^{2N-1} \exp\left(i \frac{\pi k j}{N}\right) \frac{1 + (-1)^{l+j}}{2} \left\langle \frac{l + j}{2} \right| \times \hat{\rho}(t) \left| \frac{l - j}{2} \right\rangle , \qquad (13)$$

where integers l and k label the quantized momentum $P = (\hbar/2)l = \pi l/N$ and coordinate $X = \pi k/N$ of the system. We note that for the considered case of torus (as well as for the case of cylindrical phase space) the momentum is a multiple of $\hbar/2$ but not $\hbar$. Therefore, the grid has the size $2N \times 2N$. For our purpose it is more convenient to consider the "modified" Wigner function $\tilde{w}(k, l, t) = (1/4)[w(k, l, t) + w(k + N, l, t) + w(k, l + N) + w(k + N, l + N)]$. This function equals zero for odd l and k. Thus $\tilde{w}(X, P, t)$ is defined on the grid of the size $N \times N$ with $X = 2\pi k/N$ and $P = 2\pi l/N$ ($k, l = 1, \ldots, N - 1$). In the classical approach this "modification procedure" denotes that we consider the system dynamics on modulo π instead of modulo 2π.

In our numerical simulation we choose $U(x) = [K(x^2/2 - \pi x)] \mathrm{mod} 2\pi$ in the Hamiltonian (12). In this case the classical dynamics is homogeneously unstable with the Lyapunov exponent $\lambda = \ln\{(2 + K)/2 + [(2 + K)^2/4 - 1]^{1/2}\}$. A remarkable feature of the quantum standard map on the torus with $U(x)$ of the form chosen is that it can display a specific interference pattern for integer K [12]; namely, for integer K the Wigner function equals zero everywhere except N nodes of the grid. This case is illustrated by the first row in Fig. 1, where $\tilde{w}(X, P, t)$ for $t = 1, 2, 3$ is shown for $K = 2$ and $\hbar = 2\pi/512$. (The eigenstate $|0\rangle$ was chosen as an initial condition. In the classical case it corresponds to an ensemble of the particles uniformly distributed over X with $P = 0$.) We would like to note that this interference pattern is not typical for the quantum standard map and is sensitive to any perturbation (see Fig. 1, second row). However, because of its transparent character, it is ideally suited for the study of the transition between quantum and classical dynamics.

The third row in Fig. 1 shows the Wigner function for an open system or, what is the same, the Wigner function averaged over the stochastic process. The interaction

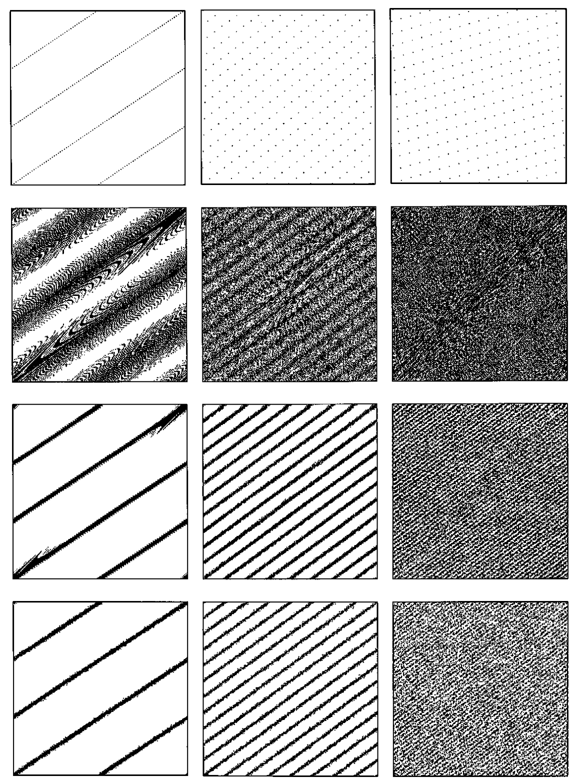

FIG. 1. Comparison between the quantum ($\hbar = 2\pi/N$, $N = 512$) and classical dynamics of the standard map on the torus for $K = 2$. A quarter of the phase space ($0 \leq X, P < \pi$) is shown. For the Wigner function the following symbolic representation is used: big dots correspond to the positive value of $\tilde{w}(X, P, t)$ [being together, big dots give black color in the figures], small dots denote negative value (they give gray color), and the absence of the dots (white color) corresponds to the value of the Wigner function near zero [$|\tilde{w}(X, P, t)| < \varepsilon$, $\varepsilon \sim N^{-2}$]. First row, unperturbed dynamics; second row, sample evolution for one realization of the stochastic process with $\epsilon = 0.05$; third row, the Wigner function averaged over the stochastic process; fourth row, dynamics of an ensemble of the classical particles for $\epsilon = 0.05$.

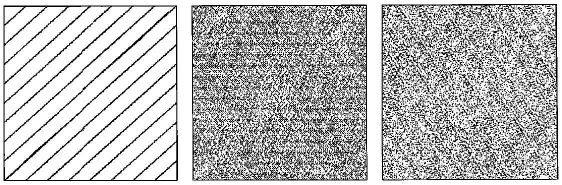

FIG. 2. Dynamics of the Wigner function averaged over the stochastic process for $\hbar = 2\pi$, $\epsilon = 0.05$, and $K = 8$. A sign of the quantum interference pattern is clearly seen for $t > 1$.

operator $\hat{V}$ in Eq. (4) was chosen in the form $V(x) = [x - \pi]\,\mathrm{mod}2\pi$; i.e., $V(x)$ is proportional to the first derivative from $U(x)$. (Such a form of the interaction operator is a general one for the case of a mesoscopic particle with few internal degrees of freedom [4].) A good correspondence with the classical dynamics (fourth row) is seen, and we have found this correspondence for an arbitrary large time available for the computer.

In the numerical simulation presented in Fig. 1 we chose the parameter ϵ on the border of validity of the condition (11). Figure 2 shows the dynamics of the Wigner function for the same values of ϵ and $\hbar$, but $K = 8$. Now the Lyapunov exponent is approximately two times larger than for the case $K = 2$, and the condition (11) should be violated. In fact, one can see the sign of the interference pattern discussed above (the classical distribution function would look completely uniform for $t > 1$) and, thus, Fig. 2 confirms our theoretical predictions.

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[1] T. Dittrich and R. Graham, Europhys. Lett. **7**, 287 (1988).

[2] M. Toda, S. Adachi, and K. Ikeda, Prog. Theor. Phys. Suppl. **98**, 323 (1989).

[3] W. H. Zurek, Phys. Today **10**, 2 (1991); Prog. Theor. Phys. **89**, 281 (1993).

[4] A. R. Kolovsky, Europhys. Lett. **27**, 79 (1994); Europhys. Lett. **29**, 655 (1995).

[5] W. H. Zurek and J. P. Paz, Phys. Rev. Lett. **72**, 2508 (1994).

[6] M. C. Gutzwiller, J. Math. Phys. **8**, 1979 (1967).

[7] G. P. Berman and G. M. Zaslavsky, Physica (Amsterdam) **91A**, 450 (1978). A simple way to estimate t_c is given in Ref. [4].

[8] E. J. Heller and S. Tomsovic, Phys. Today **7**, 38 (1993); Phys. Rev. E **47**, 282 (1993).

[9] N. Gisin and I. C. Percival, J. Phys. A **25**, 5677 (1992); Phys. Lett. A **167**, 315 (1992).

[10] We note that this result can also be obtained by using the stationary phase method to estimate the integrals in Eq. (7). [A. R. Kolovsky,"Quantum Coherence, Evolution of the Wigner Function, and Transition from Quantum to Classical Dynamics" (unpublished).]

[11] In comparison with Eq. (5), transformation (13) uses the momentum representation for the system density matrix. The formula is adopted from the paper by T. Dittrich and R. Graham [Europhys. Lett. **4**, 263 (1987)].

[12] For integer K the system discussed is known as quantum Arnold's cat map: J. H. Hannay and M. V. Berry, Physica (Amsterdam) **1D**, 267 (1980); G. H. Ristow, "A Quantum Mechanical Investigation of the Arnold's Cat Map, 1987" (unpublished); J. Ford and G. Mantica, Am. J. Phys. **60**, 1086 (1992).