Time Correlations in 1D Quantum Impurity Problems

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We develop an analytical approach using form factors to compute time dependent correlations in integrable quantum impurity problems. As an example, we obtain for the first time the frequency dependent conductivity $G(\omega)$ for the tunneling between edges in the $\nu = 1/3$ fractional quantum Hall effect and the spectrum $S(\omega)$ of the spin-spin correlation in the anisotropic Kondo model and equivalently in the double well system of dissipative quantum mechanics, both at vanishing temperature. [S0031-9007(96)00099-3]

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Several problems of great physical interest can be reformulated as impurity problems in one dimensional Luttinger liquids. These include the anisotropic Kondo problem, the double well and multiwell problem in dissipative quantum mechanics [1], and the tunneling between edges in the fractional quantum Hall effect [2–4]. Although these problems are integrable, exact results have for a long time mostly concerned static, thermodynamic quantities. A recent approach using massless scattering has also allowed the computation of dc transport properties and noise [5–7], in and out of equilibrium. Unfortunately, ac properties had so far remained inaccessible, while containing most of the open physical questions. In this Letter, we develop an approach based on "massless form factors" to compute the ac properties. We illustrate it on two simple cases, both at vanishing temperature: the frequency dependent conductivity $G(\omega)$ for the tunneling between edges in the $\nu = 1/3$ fractional quantum Hall effect and the spectrum $S(\omega)$ of the spin-spin correlation in the double well problem of dissipative quantum mechanics and in the anisotropic Kondo model.

Edge excitations in the fractional quantum Hall effect with filling fraction $\nu = 1/(2p + 1)$ are thought to be described by a chiral Luttinger liquid [2]. This allows a number of interesting theoretical predictions to be made, in particular, concerning the effect of impurities [3]. For a single impurity (as can be obtained experimentally through a constriction), one expects, based on perturbation theory and scaling arguments, the (dimensionless) dc conductance to behave as $cT^{2(1/\nu-1)}$ at low temperature and $\nu + c'T^{2(\nu-1)}$ at high temperature. The problem can be described by the Hamiltonian,

$$H = \frac{1}{2}\int_{-\infty}^{\infty} dx\left[4\pi\nu\Pi^2 + \frac{1}{4\pi\nu}(\partial_x\phi)^2\right]_R + \left[4\pi\nu\Pi^2 + \frac{1}{4\pi\nu}(\partial_x\phi)^2\right]_L + \lambda\cos[\phi_L(0) - \phi_R(0)], \tag{1}$$

where $\phi_{L/R}(x \pm t)$ are left and right moving fields moving on different edges. This model can be simplified through a change of variables,

$$\begin{aligned}\phi^e(x,t) &= \frac{1}{\sqrt{2}}[\phi_L(x,t) + \phi_R(-x,t)],\\ \phi^o(x,t) &= \frac{1}{\sqrt{2}}[\phi_L(x,t) - \phi_R(-x,t)].\end{aligned} \tag{2}$$

The even field factorizes, and after folding the real line we get a Hamiltonian on the half line with interaction at the boundary which can be rearranged in the form

$$H = \frac{1}{2}\int_{-\infty}^{0} dx\left[8\pi\nu\Pi^2 + \frac{1}{8\pi\nu}(\partial_x\phi)^2\right] + \lambda\cos[\phi(0)/2]. \tag{3}$$

This Hamiltonian is integrable, and the problem is thus "solvable" in principle. The dc conductance [5] and the dc shot noise [6,7] have been computed in the general case of a nonvanishing voltage. These results followed from combining the standard thermodynamic Bethe ansatz with a Boltzmann equation.

The problem of a two state system in a dissipative bath can also be mapped onto a boundary problem [1]. When the dissipation is Ohmic, it can described by a single spin interacting with a bath of electrons. The corresponding Hamiltonian is the anisotropic Kondo model, which is also known to be integrable,

$$H = \frac{1}{2}\int_{-\infty}^{0} dx\left[8\pi\alpha\Pi^2 + \frac{1}{8\pi\alpha}(\partial_x\phi)^2\right] + \lambda(\sigma_+e^{i\phi(0)/2} + \sigma_-e^{-i\phi(0)/2}), \tag{4}$$

where $\sigma_\pm$ are Pauli matrices. The models (3) and (4) have the same "bulk" part: a free boson (we denote the coupling ν or α by g in the general discussion to follow), but they differ significantly by the boundary interaction. They both belong to the same large class of integrable boundary field theories, and can be approached by a unified formalism [8]. The general strategy is to describe the bulk part by a basis of states that scatter in a very simple way on the boundary. To find the right basis, one can think of more general Hamiltonians that look like (3) and (4) but with an additional term of the

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form $\Lambda \int_{-\infty}^{0} dx \cos[\phi(x)]$. The bulk part is then the well known sine-Gordon model, while it can be shown that the boundary interaction does not spoil integrability [9]. The right basis for the sine-Gordon model is well known; it consists of (massive) solitons or antisolitons, and for $g \leq 1/3$, $[1/g - 2]$ breathers. These have factorized scattering, and they also scatter simply, without particle production, at the boundary. One can now take the limit $\Lambda \to 0$ to obtain a description of the free boson using massless solitons or antisolitons and breathers [10]. This is, of course, a more complicated basis than the standard plane waves, but in the latter the effect of the boundary interaction is essentially intractable, while here it is transparent. Massless particles have dispersion relation $E = p$ $[E = -p]$ for right (R) [left (L)] moving particles. Since the theory is massless, we can set the arbitrary energy scale equal to unity, and parametrize the energies by a rapidity: $E = e^{\beta}$ for solitons or antisolitons, $E = 2\sin[n\pi g/2(1-g)]e^{\beta}$ for the nth breather.

In what follows, we work in Euclidean space and choose x to be the imaginary time. This is a "modular transformed point of view," where the boundary interaction does not appear in the Hamiltonian any more, but is encoded in a boundary state $|B\rangle$, so correlators can be represented as $\langle 0|\mathcal{O}\,\mathcal{O}|B\rangle$. The boundary state has a simple expression [9] in terms of solitons or antisolitons and breathers creation operators, $Z_{\epsilon}^{*}(\beta)$, with $\epsilon = \pm, 1, \ldots, [1/g - 2]$,

$$|B\rangle = \sum_{n,\epsilon's} \int K^{\epsilon_1\epsilon_1'}(\beta_B - \beta_1)\cdots K^{\epsilon_n\epsilon_n'}(\beta_B - \beta_n) \times Z_{\epsilon_1 L}^{*}(\beta_1)\cdots Z_{\epsilon_n L}^{*}(\beta_n) Z_{\epsilon_1' R}^{*}(\beta_1)\cdots Z_{\epsilon_n' R}^{*}(\beta_n)|0\rangle\,, \tag{5}$$

where the integrals run on $[-\infty,\infty]$ and are ordered $\beta_1 < \cdots < \beta_n$. The boundary interaction is completely encoded in the matrix K^{ab}, which derives from the reflection matrix, solution of the boundary Yang-Baxter equation,

$$K^{ab}(\beta) = R_{\bar{b}}^{a}\left(\frac{i\pi}{2} - \beta\right). \tag{6}$$

The strength of the boundary interaction is encoded in an energy scale $T_B = e^{\beta_B}$. The latter is related in a nonuniversal way to the microscopic coupling, $T_B \propto \lambda^{1/(1-g)}$.

To compute a correlation function in this approach, one needs to know the matrix elements of the operators in the massless particle basis, the so-called form factors [11,12]. Away from $g = 1/2$, there are infinitely such matrix elements to compute, since the theory is truly interactive. For instance, the U(1) current acting on the vacuum can create arbitrary numbers of solitons or antisolitons pairs and breathers. There are thus two difficulties: to compute the matrix elements and to sum their contributions. In this Letter, we will show how to compute the form factors by taking appropriate limits of the massive sine-Gordon form factors. We will also show that expansions in multiparticle processes converge extremely fast, so in practice only a few terms are necessary to obtain excellent precision (10^{-3}) on the quantities of interest, and this all the way from the short distance to the large distance fixed point.

We start with the frequency dependent conductance in the problem of tunneling between edge states. Using the Kubo formula [3] and folding the system as explained above one finds $G = (g/2) + \Delta G(\omega)$, where

$$\Delta G(w_M) = \frac{1}{8\pi\omega_M L^2}\int_{-L}^{0} dx\, dx' \int_{-\infty}^{\infty} dy\, e^{i\omega_M y} \times [\langle \partial_z \phi(x,y) \partial_{\bar{z}'}\phi(x',0)\rangle + \text{c.c.}]\,, \tag{7}$$

ω_M being a Matsubara frequency, $z = x + iy$. We have used here the fact that in the current-current correlation the term $\langle \partial_z \phi \partial_{z'} \phi\rangle$ and its conjugate are insensitive to the impurity [13], contributing $g/2$ to the conductance. This is because $\partial_z \phi$ $(\partial_{\bar{z}}\phi)$ act only on R (L) particles, while the effect of the boundary is to mix L and R particles. Considering a particular term in the expansion of $|B\rangle$ with n L and n R particles, the only process with a nonvanishing amplitude in $\langle 0|\partial_z\phi\partial_{\bar{z}'}\phi|B\rangle$ is to have $\partial_{\bar{z}'}\phi$ annihilate all the n L particles and then $\partial_z\phi$ all the n remaining R particles. We thus need the form factors

$$f(\beta_1,\ldots,\beta_n)_{\epsilon_1,\ldots,\epsilon_n} = \langle 0|\partial_z\phi(0) Z_{\epsilon_1 R}^{*}(\beta_1)\cdots Z_{\epsilon_n R}^{*}(\beta_n)|0\rangle\,. \tag{8}$$

Current form factors have been computed by Smirnov [11] for the massive sine-Gordon model. Expressions for (8) can then be obtained by taking a massless limit, i.e., by sending the physical mass to zero and the rapidities to infinity to keep excitations of finite energy [12].

As examples, when $g = 1/2$, there is only a pair of solitons or antisolitons in the spectrum. The only nonzero form factor is $f_{\pm\mp}(\beta_1,\beta_2) = \mp c e^{(\beta_1+\beta_2)/2}$, with c a known normalization factor. For $g = 1/3$, a breather appears in the spectrum and the first two form factors are given by

$$f_1(\beta) = c_1 e^{\beta} \tag{9}$$

for the 1 breather and

$$f_{\pm,\mp}(\beta_1,\beta_2) = \mp c_2 \frac{\zeta(\beta_1-\beta_2)}{\cosh(\beta_1-\beta_2)} e^{(\beta_1+\beta_2)/2} \tag{10}$$

for the two soliton form factors. Here $\zeta(\beta)$ is a known function and c_1, c_2 known constants, whose expressions we will give elsewhere [14].

From (5), the form factors expansion results in the general expression

$$\langle \partial_{\bar{z}}\phi(x,y)\partial_{z'}\phi(x',y')\rangle = \int_0^{\infty} dE\, \mathcal{G}(E) \exp[E(x+x') - iE(y-y')]\,, \tag{11}$$

and from (7)

$$\Delta G(\omega) = \frac{1}{4\omega}\,\text{Im}\mathcal{G}(-i\omega)\,. \tag{12}$$

Using the previous formulas and the expression for the form factors and the boundary scattering matrices [9], we find

$$G(\omega) = \tfrac{1}{2}[1 - (T_B/\omega)\tan^{-1}(\omega/T_B)], \qquad (13)$$

for $g = 1/2$ in agreement with previous results [15]. For $g = 1/3$, there is an infinity of form factors contributing to the correlation. However, when computing the conductivity, we find a very rapid convergence with the number of rapidities; two rapidities are sufficient to get a 1% accuracy. Let us stress that this convergence is independent of the strength of the impurity, and the results are valid for the *whole* flow from small to large distances. The quantity $G(\omega)$ is plotted in Fig. 1 as a function of T_B/ω.

Some general features of $G(\omega)$ can easily be deduced from this approach. The reflection matrices of solitons or antisolitons expand as a double power series in $\exp\beta$ and $\exp[(1/g) - 1]\beta$, the reflection matrix of breathers as a power series in $\exp\beta$. This leads to a double power series in $(\omega/T_B)^{-2+2/g}$ and $(\omega/T_B)^2$ at small frequencies, $(T_B/\omega)^{2-2g}$ and $(T_B/\omega)^2$ at large frequencies. Therefore, as first argued in [17], at low frequencies, $G(\omega)$ goes as ω^2 for $g < 1/2$ and $\omega^{-2+2/g}$ for $g > 1/2$.

The same method can be applied to compute the spin-spin correlation $C(t) = \frac{1}{2}\langle[\sigma(t), \sigma(0)]\rangle$ or its Fourier transform conventionally denoted $\chi''(\omega)$ in the two state problem [1]. A difficulty arises at first sight because the massless scattering description of the anisotropic Kondo problem involves only the massless sine-Gordon particles and no spins (physically, this is because this description is based on the large distance limit of the theory where the spin is screened). However, using the fact that spin flips, which are induced by $\sigma_\pm$, are coupled with insertions of vertex operators at the boundary, one can relate $C(t)$ to the current correlator, and get the expression

$$\chi''(\omega) = \frac{1}{(2g\pi)^2}\frac{1}{\omega^2}\,\mathrm{Im}[\mathcal{G}(-i\omega, T_B) - \mathcal{G}(-i\omega, 0)], \qquad (14)$$

where $\mathcal{G}$ is the Fourier transform of the L-R current correlator defined in (11). The only difference with the conductance problem is in the boundary interaction. The reflection matrix for the solitons are now given by [10]

$$R_\pm^\pm = \tanh\left(\frac{\beta}{2} - \frac{i\pi}{4}\right), \qquad R_\mp^\pm = 0, \qquad (15)$$

and for the breathers we find [14]

$$R_m^m = \frac{\tanh[\beta/2 - i\pi m/4(1/g - 1)]}{\tanh[\beta/2 + i\pi m/4(1/g - 1)]}. \qquad (16)$$

The computations are then done along the same lines as before. For $g = 1/2$, the free or Toulouse point, the previously known result,

$$\chi''(\omega) = \frac{1}{\pi^2}\frac{4T_B^2}{\omega^2 + 4T_B^2} \times \left[\frac{1}{\omega}\ln\left(\frac{T_B^2 + \omega^2}{T_B^2}\right) + \frac{1}{T_B}\tan^{-1}\frac{\omega}{T_B}\right], \qquad (17)$$

is recovered. For a whole domain of $g \in [0.25, 0.6]$, the form factor expansion gives again very precise results for $\chi''(\omega)$ with only the first two terms, for all strengths of the impurity coupling. Let us give as an example the explicit expressions for $g = \frac{1}{3}$. The contribution from the breather is

$$\frac{c}{\omega}\,\mathrm{Re}\left[R_1^1\left(\ln\left(\frac{\omega}{\sqrt{2}T_B}\right)\right) - 1\right], \qquad (18)$$

with $c = -0.141$ and the contribution for the two soliton form factors is given by

$$\frac{c'}{\omega}\,\mathrm{Re}\int_{-\infty}^{0} d\beta\,\frac{|\zeta(\beta - \ln(1 - e^\beta))|^2}{\cosh^2[\beta - \ln(1 - e^\beta)]}\,e^\beta \times \left\{R_+^+\left(\beta + \ln\left(\frac{\omega}{T_B}\right)\right)R_+^+\left(\ln\left[(1 - e^\beta)\frac{\omega}{T_B}\right]\right) - 1\right\}, \qquad (19)$$

with $c' = -0.0451$ and the function $\zeta(\beta)$ given in [11]. In Fig. 2 we show the function $S(\omega) = \chi''(\omega)/\omega$ for $g = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{3}{5}$ at $T_B = 0.1$. From the scaling form $S(\omega) = (1/\omega^2)F(\omega/T_B)$, the results for all T_B are recovered. We observed with very good precision that the "quasiparticle peak" [1] disappears at $g = \frac{1}{3}$ (instead of the sometimes conjectured $g = \frac{1}{2}$), but we have no analytical proof of this result. Physically this means that

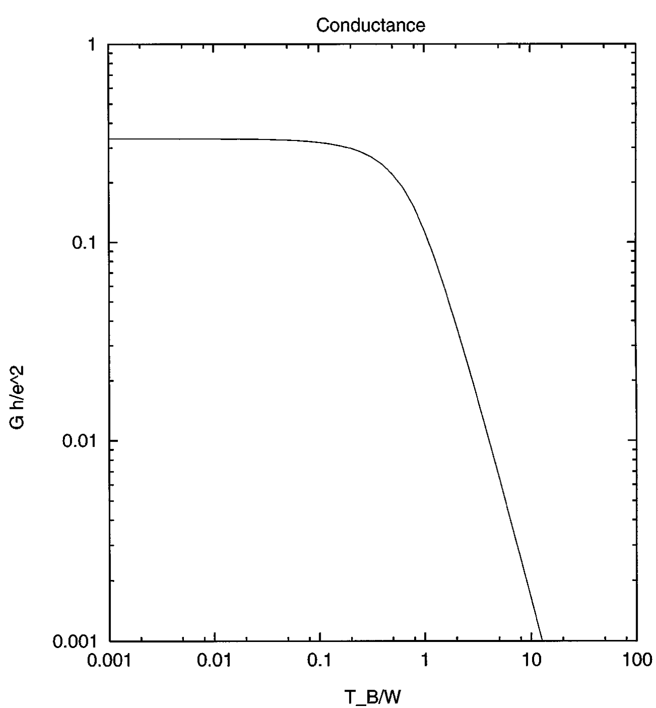

FIG. 1. Frequency dependent conductivity at $T = 0$, $g = 1/3$.

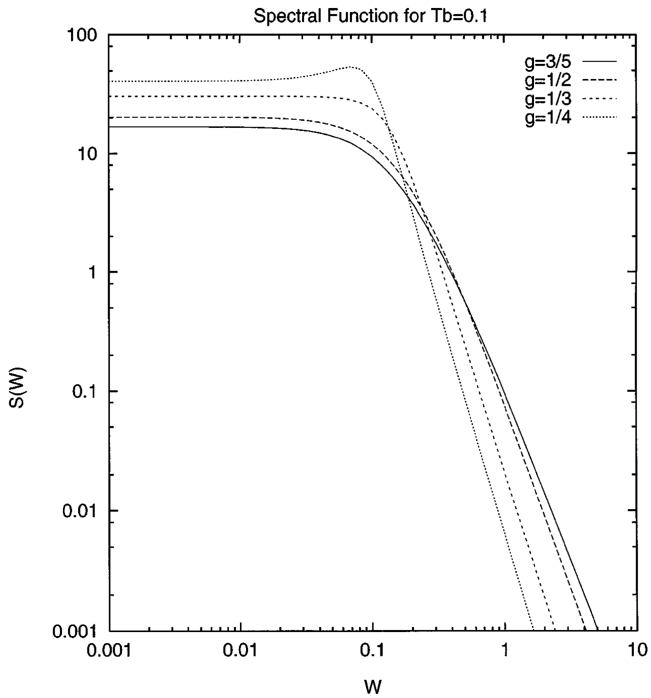

FIG. 2. Spectral function $S(\omega)$ for different values of g.

the crossover between the oscillating behavior and the damped one is at $g = \frac{1}{3}$ at zero temperature. This is in agreement with recent renormalization group numerical results [18].

It is easy to check from the form factor expansion that for all g the following relation holds:

$$\lim_{\omega\to 0} \frac{\chi''(\omega)}{\omega} = \frac{1}{\pi^2 T_B^2 g}, \tag{20}$$

which leads to the large time asymptotics $C(t) \approx -(\sqrt{\pi}/eg)[1/(\pi T_B t)^2]$.

At small frequency, $S(\omega)$ expands as a power series in $(\omega/T_B)^{2n}$ for any g. At large frequency, it expands as a double series in $(T_B/\omega)^2$ and $(T_B/\omega)^{2-2g}$. As a consequence, $C(t) \propto t^{2-2g}$ at small times.

In conclusion, we have shown that the description of quantum impurity problems based on (i) massless integrable boundary field theories, (ii) boundary states, and (iii) form factors allows an efficient computation of the time dependent properties. So far, these had been completely inaccessible except by Monte Carlo simulations, and by various approximations which were recently proven unreliable [16]. Our goal here was to present the technique and some results. Details will be provided in a more extensive paper. We also expect to be able to extend the method to the case where there is a voltage (a bias) and to the finite temperature case. This will involve a combination of both the form factors and the TBA approach. Also, the careful reader might wonder why the computation of ac properties requires form factors, while dc properties have been successfully computed so far by analogy with the free case. The reason is that dc properties see only the charge Q, i.e., the x integral of the current, and Q can be shown to act purely diagonally on multiparticle states. See [14] for more details.

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