Mechanism for Global Optimization of River Networks from Local Erosion Rules

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We show that landscapes and their drainage networks evolve under erosion rules to obey a variational principle. Starting from hydrodynamics we first find by separation of variables a general relationship between erosion and a modified slope-area law. This law encompasses previous work and observations, and we confirm it by simulation. Secondly we show that this corresponds to a solution of a variational minimization model, generalized from Rodriguez-Iturbe *et al.* Thus a mechanism by which the local process of erosion acts to globally optimize a river network is produced. [S0031-9007(96)00098-1]

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Empirical observations on natural rivers by Leopold and Maddock [1] and Leopold [2] revealed power-law relationships between the slope (s), width (w), depth (d), velocity (v), and discharge (Q) of a channel, measured at the same time for different locations amongst river networks in the United States:

$$s \propto Q^{-0.5}, \qquad w \propto Q^{0.5}, \qquad d \propto Q^{0.4}, \qquad v \propto Q^{0.1}. \tag{1}$$

The first of these is an observation of the slope-area law. A theoretical framework was put forward by Rodriguez-Iturbe *et al.* [3] that was able to account for some of these observations. Using two unproven principles an expression was derived for the total rate of expenditure of the mechanical potential energy of all the water in the network (E):

$$E = \sum_{\text{landscape}} Q^{0.5}. \tag{2}$$

The unsubstantiated suggestion was then made that natural river networks evolve in such a way as to make E a global minimum. Computed networks that obey this rule are called optimal channel networks, and these networks have been found to obey Hack's law, Horton's laws, Moon's law, Melton's law, Strahler ordering, and the drainage-basin area law to the same extent as natural networks [4–8]. Different global minimization principles were also suggested by Yang, Song, and Davies and Sutherland [9–11], but in each case a full justification for the method proved difficult to obtain.

Rinaldo *et al.* [12,13], and latterly Sun, Meakin, and Jossang [14], performed a direct minimization of E by computer simulation, using a simulated annealing approach. In this model, a channel network was placed on a square lattice such that the water present at each site in the lattice could drain through the network to the perimeter of the lattice. This network was then rearranged, until a drainage network with a minimum value of E was found. This value was remarkably constant, using a wide variety of initial conditions. The identification $s \propto Q^{-0.5}$ was then made [15], enabling a value of height to be allotted to each site in the lattice, and it was discovered that the scalar field of heights was continuous over basin boundaries.

In this Letter basic hydrodynamics is used to establish an erosion equation, which is numerically shown to erode the landscape to give steady drainage networks. We then show analytically that this implies a particular generalization of the slope-area law. This law is then revealed to be equivalent to the minimization of a global quantity L, which explains how local erosion can act to minimize a global quantity, in analogy to Hamilton's principle of least action in mechanics [16]. The equivalence between the slope-area law and minimization of E forms a specific case of this mapping.

We begin with four well-known hydrodynamic principles:

$$Q = vdw, \quad \tau \approx \rho g s d, \quad v \approx (gsd)^{1/2}, \quad w \propto d, \tag{3}$$

where τ = bed stress; a different homogeneous relationship between these quantities would require only minor modifications to our work. The local rate of erosion of the landscape height h with time t is taken to be a homogeneous function of the variables discussed above, $\partial h/\partial t \propto s^{a_1} Q^{a_2} v^{a_3} w^{a_4} d^{a_5} \tau^{a_6}$, where a_{1-6} are constants. Since the six variables are already related by four equations, we can express $\partial h/\partial t$ in terms of any two of them. In particular, the following two forms are equivalent and consistent:

$$\partial h/\partial t \propto \tau^{2.5\gamma} Q^{\epsilon} \propto s^{2\gamma} Q^{\gamma+\epsilon}. \tag{4}$$

The second form of Eq. (4) is similar to the model of Inaoka and Takayasu [17] and can readily be simulated by computer. Each site on a square lattice is given a random initial height and unit precipitation, which drains to the site's lowest neighbor, forming a network of streams and lakes, which are allowed to fill up until they overflow. At each site the height is eroded according to Eq. (4). The drainage directions and lakes are then computed for the new heights, and the process repeated.

During these simulations it was observed that the number and size of the lakes decreased as their outlets were eroded until, after ~75% of the material initially present had been eroded, there were no more lakes present. The drainage network then became constant while the heights continued to decrease. This observation is extremely robust to variations in initial conditions and exponents, although the shape and characteristics of the final channel network are subject to change (see Fig. 1).

The erosion equation is

$$\partial h(\vec{x},t)/\partial t \propto s(\vec{x},t)^{2\gamma} Q(\vec{x})^{\gamma+\epsilon}, \tag{5}$$

and the evolution of this must be such that no site changes its direction of drainage. The simplest possibility is that $h(\vec{x},t)$ approaches a separable variable form

$$h(\vec{x},t) = g(t)\eta(\vec{x}). \tag{6}$$

This is fully consistent with Eq. (5) if and only if

$$\frac{dg/dt}{g(t)^{2\gamma}} = \text{const} = \frac{|\vec{\nabla}\eta|^{2\gamma}}{\eta(\vec{x})} Q(\vec{x})^{\gamma+\epsilon}. \tag{7}$$

Integrating the first equality in Eq. (7) gives $g(t) \propto t^{1/(1-2\gamma)}$, and the second equality gives a general relation between slope, height, and discharge. This analysis provides two propositions that can readily be tested by computer, using the model descibed earlier to simulate erosion in the long time regime. These are as follows: (i) at a particular site

$$h \propto t^{1/(1-2\gamma)}; \tag{8}$$

(ii) at a particular time,

$$sh^{-1/2\gamma} \propto Q^{-(\gamma+\epsilon)/2\gamma}. \tag{9}$$

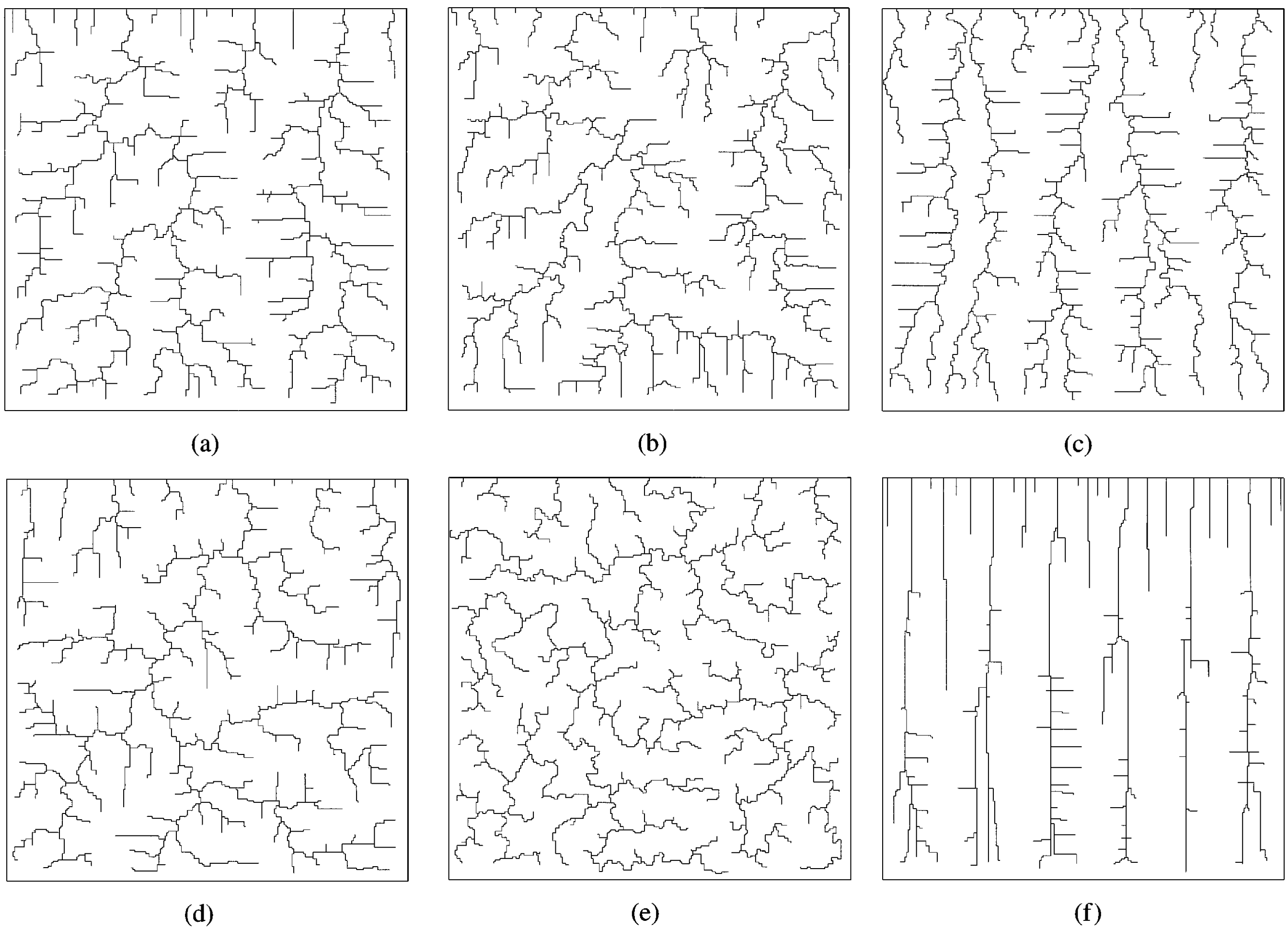

FIG. 1. Six 250 × 250 stream networks produced by simulation of Eq. (4), showing channels draining ≥100 sites. (a)–(d) have exponents $\gamma = 1$, $\epsilon = -0.2$, but different initial landscape heights. (a) $h_i = 20\,000 + 0.1p_i$, (b) $h_i = 20\,000 + 1000p_i$, (c) $h_i = 20\,000x/250 + 1000p_i$, where x is the distance of the site from the seashore, and $P = \{p_i\}$ is a set of random numbers from 0 to 1. (d) The same as (a) with different random numbers. Comparing (a) with (b) shows that varying the noise has no systematic effect, and (c) shows that ramping the initial landscape changes the appearance because of downstream bias. (e) and (f) have the same initial conditions and random numbers as (d), but different exponents. For (e), $\gamma = 1, \epsilon = 0.9$, and (f) has $\gamma = 1, \epsilon = -0.8$. Networks (a), (b), and (d) give good agreement with empirical data for the drainage area distribution (see Fig. 2), Hack's law exponent, Strahler bifurcation ratio, and Horton's ratios. Note how in networks (f), (d), and (e) the sinuousity increases from unrealistically low to unrealistically high as $(\gamma + \epsilon)/2\gamma$ increases from 0 to 1, implying that there is a value of $(\gamma + \epsilon)/2\gamma$ that will accurately reproduce natural river networks.

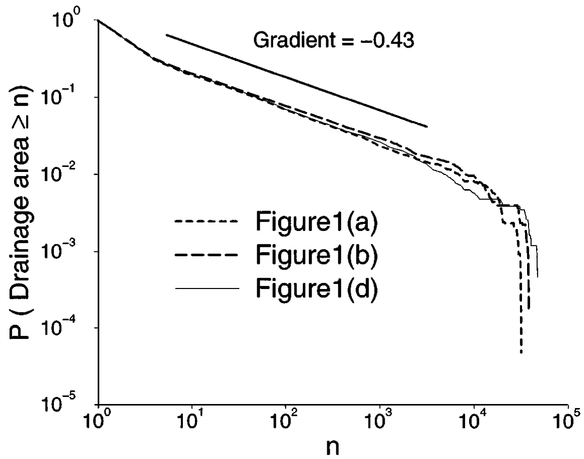

FIG. 2. Drainage area distribution for networks (a), (b), and (d) from Fig. 1. The straight line on this graph indicates that the drainage area has a power-law distribution, and the gradient agrees well with empirical data [18].

On a continuous space of $\vec{x}$, such as a natural landscape, Eq. (9) can be written in vector form as

$$\vec{\nabla}(h^{1-1/2\gamma}) \propto Q^{-(\gamma+\epsilon)/2\gamma}\hat{Q}\,, \tag{10}$$

where Q is the magnitude of the vector $\vec{Q}$, and $\hat{Q}$ is a unit vector in the direction of $\vec{Q}$. However, for the discrete lattices used in computer simulations, Eq. (10) is subject to the approximation

$$\Delta h(h^{-1/2\gamma}) \approx \Delta(h^{1-1/2\gamma})/(1 - 1/2\gamma)\,. \tag{11}$$

The effect of this approximation is negligible for the simulations used in this Letter, as can be seen along with the satisfaction of Eq. (10) from Fig. 4 and Table II. The satisfaction of Eq. (8) is illustrated in Fig. 3 and Table I.

We now show that Eq. (10) can be interpreted as the Euler-Lagrange equation corresponding to the condition

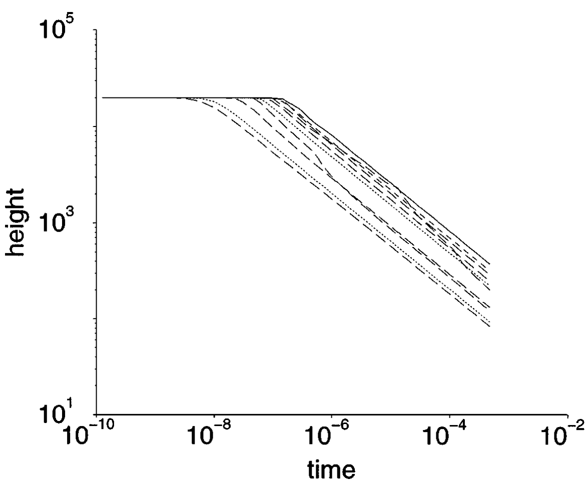

FIG. 3. Scaling of height with time. Example of a graph showing height vs time for ten randomly chosen sites from a 250 × 250 lattice, with $2\gamma = 3$ and $\gamma + \epsilon = 1.5$. Table I gives predicted and observed gradients; the shape of the graph is very robust with respect to changes in 2γ and $\gamma + \epsilon$.

TABLE I. Results from graphs of height vs time for different runs of the simulation program. See Fig. 3.

2γ	$\gamma + \epsilon$	Expected gradient	Observed gradient
1.1	3	−10	−9.79
1.1	5	−10	−9.85
1.4	1.4	−2.5	−2.49
2	0.3	−1	−0.99
2	2	−1	−1.00
3	1.5	−0.5	−0.50

that

$$L = \int_{\vec{x}} Q^{(\gamma-\epsilon)/2\gamma}\,d\vec{x} \tag{12}$$

is made stationary, subject to the constraint that all rainfall drains out of the landscape through the channel network, which can be expressed mathematically as

$$\vec{\nabla}\cdot\vec{Q}(\vec{x}) = R(\vec{x})\,, \tag{13}$$

where $R(\vec{x})$ is the precipitation rate specified at each site $\vec{x}$. After introducing $\lambda(\vec{x})$ as a Lagrange multiplier field, the constrained variational problem is equivalent to the unconstrained variation

$$\frac{\delta}{\delta Q}\int_{\vec{x}}\{Q^{(\gamma-\epsilon)/2\gamma} + \lambda[\vec{\nabla}\cdot\vec{Q}(\vec{x}) - R(\vec{x})]\}\,d\vec{x} = 0\,. \tag{14}$$

This gives the Euler-Lagrange equations

$$\vec{\nabla}\lambda - [(\gamma - \epsilon)/2\gamma]Q^{-(\gamma+\epsilon)/2\gamma}\hat{Q} = 0\,, \tag{15}$$

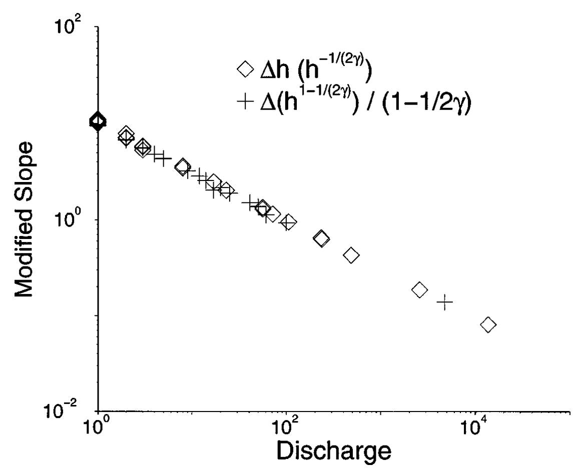

FIG. 4. Scaling of modified slope with discharge. Example of a graph showing both $\Delta(h^{1-1/2\gamma})/(1 - 1/2\gamma)$ and $\Delta h(h^{-1/2\gamma})$ against discharge for 40 randomly chosen sites from the same run of the simulation program as Fig. 3. The effect of the approximation in (11) is clearly negligible. The gradient $= -0.5$, which is equal to $-(\gamma + \epsilon)/2\gamma$, as predicted by Eq. (10). The shape of this graph is also very robust to variation in 2γ and $\gamma + \epsilon$ (see Table II).

TABLE II. Results from graphs of $\Delta(h^{1-1/2\gamma})$ vs Q for different runs of the simulation program. See Fig. 4.

2γ	$\gamma + \epsilon$	Expected gradient	Observed gradient
1.1	3	-2.72	-3.01
1.1	5	-4.55	-4.44
1.4	1.4	-1	-1.13
2	0.3	-0.15	-0.15
2	2	-1	-1.07
3	1.5	-0.5	-0.51

using appropriate boundary conditions: either $Q_{\text{boundary}} = 0$ (corresponding to a mountainous boundary) or $\lambda_{\text{boundary}} = 0$ (corresponding to a coastline, beyond which the constraint is undefined).

Equations (15) and (10) are identical in form, enabling the identification $\lambda(\vec{x}) \propto h(\vec{x})^{1-1/2\gamma}$ to be made, where the constant of proportionality is a function of time. This result implies that the necessary and sufficient condition for L to be stationary, subject to the constraint of Eq. (13), is equivalent to the general relation between modified slope and discharge derived from local erosion rules, and therefore a landscape eroded by Eq. (5) will exist in a state of stationary L. Numerically we have monitored L after the lakes have disappeared, and it invariably decreases, usually monotonically. We can show analytically that stream capture across basin boundaries that lowers L must eventually occur [19], implying that erosion minimizes L with respect to basin boundary motion.

Finally we consider the case when the rate of erosion is primarily and strongly dependent on τ, i.e., $\gamma \gg \epsilon$ and $\gamma \gg 1$ in Eq. (5). Then L defined by Eq. (12) becomes a continuous version of E, and correspondingly Eq. (10) reduces to

$$\vec{\nabla} h \propto Q^{-0.5} \hat{Q} \,, \tag{16}$$

which is the vector form of the slope-area law. Therefore any landscape that obeys the slope-area law must have a stationary value for E. This forms a specific example of the above general global optimization principle.

We have shown the direct equivalence of a steady state under erosion models and conformance to a variational principle. The quantity minimized (L) only corresponds to something similar to energy dissipation *after* minimization, when the landscape has settled into the steady state in which the modified slope-area law holds. Our work clearly establishes this procedure, but a more physical interpretation of L remains to be found.

In our arguments we have mixed results from discrete and continuous space versions of the models. Subject to the differencing approximation (11), it is possible to present all the arguments and obtain the same results for the strictly discrete lattice version. For the strictly continuous case, there are difficulties if instead of using $w \propto d$, we attempt to let the erosion equation select both width and depth of any channels formed. This problem has already been addressed by the authors [19].

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