Control of Patterns in Spatiotemporal Chaos in Optics

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We propose an algorithm for control of spatiotemporal chaos in partial differential systems based on the idea of stabilization of unstable periodic patterns embedded in spatiotemporal chaotic states. This algorithm, using time- and space-dependent feedback, has been successfully demonstrated through our numerical analysis in controlling unstable roll patterns in a transversely extended three-level laser. [S0031-9007(96)00062-2]

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Since the first suggestion of controlling chaos [1] and in particular the proposal of a control concept utilizing feedback to stabilize an unstable periodic orbit or fixed point embedded in the chaotic state [2], there has been prolific activity in the area of control of chaos across many disciplines [3]. Based on this concept of feedback control, various other approaches have been developed where emphasis has been given to algorithms which are more readily implemented in practical systems, in particular, those utilizing occasional proportional feedback [4] and continuous feedback [5]. These algorithms have been successfully implemented in spatially restrained low-dimensional chaotic systems. For spatially extended systems, spatial coherence collapses with the emergence of spatiotemporal chaos; the system then becomes infinitely dimensional. There is therefore little hope of controlling these systems by adopting the above algorithms. First efforts toward control of spatiotemporal chaos have been recently made in a one-dimensional array of chaotic elements by using a coupled feedback control approach [6–7]. The scheme developed in Ref. [6] is an extension of the Ott-Grebogi-Yorke (OGY) algorithm [2] for a single element in which the effect of coupling of the elements is accounted for in the feedback. This approach cannot, however, be readily applied to control of spatiotemporal chaos in systems comprising continuously extended media, since they are intrinsically globally coupled and infinitely dimensional. Control of these systems remains an outstanding challenge and calls for new approaches. In this Letter, we propose an algorithm for the control of spatiotemporal chaos in such systems based on the idea of stabilization of unstable periodic patterns (UPPs) embedded in spatiotemporal chaos. The implementation of this algorithm, by using small time- and space-dependent feedback to perturb a variable of the systems, has been successfully demonstrated through our numerical simulations in controlling spatiotemporal chaos in a transversely extended three-level laser.

Spatiotemporal chaos occurs when different types of motion, excited in local regions in an extended system, interact to destroy the spatial coherence of the system concurrent with the onset of temporal chaos. This phenomenon in continuous physical systems is described by partial differential equations. While the transition from coherence to spatiotemporal chaos has yet to be characterized by global quantitative laws, certain normal mode equations have shown that such a chaotic state in a spatiotemporal context underlies different unstable periodic patterns, e.g., rolls and hexagons and in this sense it is analogous with temporal chaos, which comprises many unstable periodic orbits embedded in the chaotic attractor. In optics, spatiotemporal chaos (commonly referred to as optical turbulence when the chaos is fully developed in both space and time) is attributed to the interaction, through diffraction, of optical fields excited by local dipoles and has recently been studied both theoretically and experimentally in passive nonlinear and laser systems [8–12]. A general 2D model description in optics can be given as

$$\partial \mathbf{q}/\partial t = \mathbf{N}(\mathbf{q}, \mu) + iD\nabla_\perp^2 \mathbf{q}\,, \tag{1}$$

where $\mathbf{q}$ is a set of vector variables, $\mathbf{N}$ a nonlinear function, t time, and $\nabla_\perp^2$ the transverse Laplacian. μ is the control parameter of the system and D is the matrix of diffractive coefficients. The control algorithm proposed in this Letter follows a feedback control strategy under which a feedback signal $f(x,y,t)$, which is derived from one of the above variables $\mathbf{q}$, say q_1, is assigned to perturb this variable. This feedback is designed to underly the signature of a targeted UPP in the spatiotemporal chaotic state, so it tends to vanish when the control is achieved, the feedback then being synchronized with the targeted pattern. As a demonstration of this control, we consider this targeted UPP to be an unstable roll state; a state with periodic structure in both time and space. Such periodic features suggest an algorithm with feedback of the form

$$\begin{aligned} f(x,y,t) = {} & c_1[q_1(x,y,t-t_0) - q_1(x,y,t)] \\ & + c_2\{[q_1(x+x_0,y,t) - q_1(x,y,t)] \\ & + [q_1(x,y+y_0,t) - q_1(x,y,t)]\}\,, \end{aligned} \tag{2}$$

where c_1 and c_2 are proportionality coefficients, t_0 is the period, in time, of the desired roll state in local regions, and $x_0 = 2\pi/k_x, y_0 = 2\pi/k_y$ are the characteristic lengths of

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the rolls in x and y directions in the transverse space, respectively, where k_x and k_y are the corresponding wave-vector components. From these relations and $|\mathbf{k}|^2 = k_x^2 + k_y^2$ we have $1/r_0^2 = 1/x_0^2 + 1/y_0^2$, where $r_0 = 2\pi/|\mathbf{k}|$. The feedback signal is imposed on the system continuously in time, controlling the time evolution of the patterns in local regions through a time-delayed coupling and organizing the spatial distribution in the same time through spatial network coupling in the transverse domain. The characteristic lengths, t_0 and r_0, of the desired UPP are specified in Eq. (2) in such a way that $f(x, y, t)$ tends to vanish when the chaotic state is synchronized to this UPP. We note that, in the limit of spatial homogeneity, the above control algorithm reduces to that in Ref. [5].

In demonstrating this control strategy we consider a transversely extended and coherently optically pumped three-level laser, which has recently been extensively investigated for pattern formations and optical turbulence [13]. The resonant and single mode lasing wave equation describing such a system in the presence of control feedback is given as

$$\frac{\partial E}{\partial t} = -\sigma E + gP_2 + ia\nabla_\perp^2 E + f(x, y, t), \quad (3)$$

where E is the slowly varying electric field amplitude of the laser emission, σ the cavity damping constant, g the unsaturated gain of the laser medium, and a the diffraction parameter which can be set to unity by rescaling the transverse coordinates. P_2 is the normalized density matrix element of the polarization of the lasing transition. This, together with other density matrix elements, is described by the Bloch equations generalized to a three-level system [14]. Parameters in the material equations are A, the external pump strength, and b, the ratio of energy relaxation (γ) to dipole dephasing (Γ) rates of the medium. The stability of both homogeneous steady-state $E = E_s$ and traveling wave solutions $E = E_{\text{tw}} \exp[i(\omega_{\text{tw}} t + \mathbf{k}_{\text{tw}} \cdot \mathbf{r})]$ against weak inhomogeneous perturbations has been numerically analyzed and shown to be broken through either a Hopf or static bifurcation, depending on the parameters of the system. Numerical simulations have shown optical turbulence to occur in the region where homogeneous solutions are unstable and undergo a defect-mediated scenario from a roll solution, as shown in Fig. 1. The rolls observed here have periodic structures in both real and imaginary parts of the laser field and its intensity and are distinct from the traveling wave solutions for which the intensity is uniform. Our analysis shows that such a roll state has a spatial wave number k_A corresponding to the largest growth rate of the uniform solution against inhomogeneous perturbation. We note, however, that contrary to the traveling waves the roll state possesses two independent frequencies in time in both the real and imaginary parts of the laser field, the roll frequency ω_{rol}, and the reference frequency ω_{ref}, the values of which depend on the parameters of the system. They are related by $\omega_A = \omega_{\text{rol}} - \omega_{\text{ref}}$, where ω_A is the characteristic frequency corresponding to k_A and the relation $\omega_A \sim \omega_{\text{rol}} \gg \omega_{\text{ref}}$ is found for all the parameters investigated. The existence of the reference frequency and its role in instabilities have been previously investigated by many authors in laser systems [15–17].

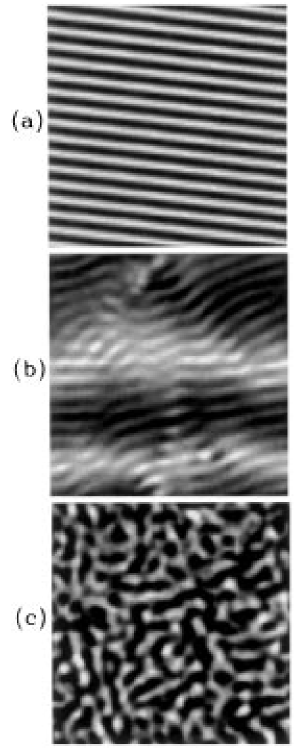

FIG. 1. Defect-mediated turbulence of Re(E) on increasing the control parameter A: (a) $A = 2.48$, a roll state with $k_A = 1.79$, $\omega_{\text{rol}} = 2.9$, and $\omega_{\text{ref}} = 0.079$; (b) $A = 2.514$, defected rolls; and (c) $A = 2.52$, an optical turbulent state. The other parameters of the system are $g = 52$, $\sigma = 1.3$, and $b = 0.4$.

To control roll patterns in an optical turbulent state of the laser system described by Eq. (3), we use the real part of the laser field as the control variable and introduce the feedback, defined in Eq. (2), to perturb this variable. The rest of the equations in the laser model description remain unchanged. Our target is a roll state, similar to that shown in Fig. 1(a), which for the parameter set used in Fig. 1(c) is unstable and embedded in the turbulent state. The characteristic wave number and frequency of the roll state for this parameter set is $k_A = 1.80$ and $\omega_{\text{rol}} \sim \omega_A = 2.90$, which are determined from linear stability analysis. The feedback $f(x, y, t)$ in our simulations applies to each grid point, total 64×64, in a square in the transverse plane of width $32\pi/k_A$. The orientation of the roll pattern is controlled through the choice of k_x and k_y (or x_0 and y_0), under the condition that the relation $k_A^2 = k_x^2 + k_y^2$ stands. Figure 2 shows the time evolution of the pattern, from an initial turbulent to a final roll state. Once the control feedback is applied, the turbulent

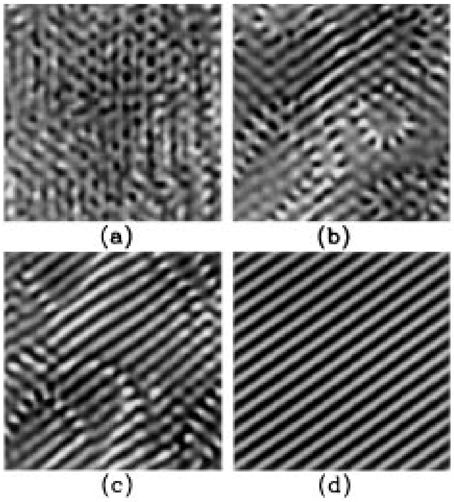

FIG. 2. Time evolution of the transverse field distribution [Re(E)] when the feedback control is applied; (a)–(d) correspond to $t = 10,\ 40,\ 80,$ and 200. The control parameters are $c_1 = c_2 = 0.92,\ t_0 = 2.17,\ x_0 = 6.11,$ and $y_0 = 4.36$, while the parameters of the system are the same as those in Fig. 1(c). 64×64 grid points in the transverse plane of width 55.8 ($32\pi/k_A$) are used in the simulation.

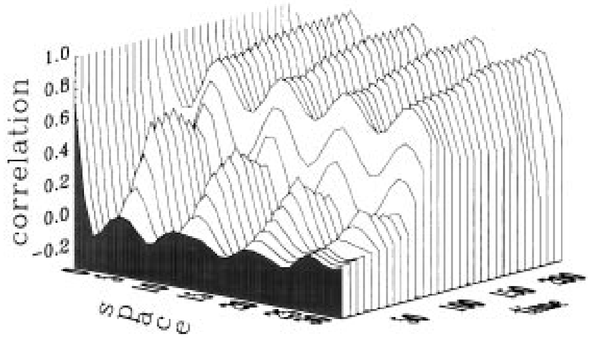

FIG. 3. Correlation function vs time in the evolution process from turbulence to rolls. The spatial unit is the number of the grid points. The parameters, transverse grid points, and size are set to be the same as those in Fig. 2.

pattern reorganizes itself to line up in the directions along with and perpendicular to **k**, $\hat{r}_{\parallel}$ and $\hat{r}_{\perp}$, though without clearly preferred orientation, resulting in a lattice structure in the turbulent background [Fig. 2(a)]. The pattern continues evolving in time, and in local regions preferred orientations emerge from the lattice structure, giving rise to localized roll structures in either $\hat{r}_{\perp}$ or $\hat{r}_{\parallel}$ directions [Fig. 2(b)]. Different orientations in small local regions then complete and merge through the coupling provided by the feedback. As the winner takes over, a preference of orientation is clearly established, as shown in Fig. 2(c), which eventually synchronizes all the wave fronts in one direction [Fig. 2(d)]. The controlled periodic pattern, as desired, indeed shows the spatial period $r_0 = 2\pi/k_A$. The feedback distribution in the transverse domain also reveals a roll structure, mimicing that of the real part of the laser field. We note, however, that for a given k_x and k_y there are still two possible orientations for the eventual roll pattern formation, either $\hat{r}_{\parallel}$ or $\hat{r}_{\perp}$, the dominance of one or the other depending on the initial condition of the system when the feedback control is applied. In quantifying the process of spatiotemporal organization of a roll pattern within the turbulence, we have examined the time evolution of the spatial correlation function of the real part of the laser field. This is defined as

$$\zeta(\rho, t) = \langle E_r(\mathbf{r}, t) E_r(\mathbf{r} + \rho\hat{\mathbf{x}}, t)\rangle / \langle E_r(\mathbf{r}, t) E_r(\mathbf{r}, t)\rangle, \tag{4}$$

where $\langle \cdots \rangle$ is taken over all possible spatial positions. Initially, $\zeta(\rho, t)$ shows typical behavior (Fig. 3, $t = 0$) of a turbulently distributed pattern, e.g., a sharply exponential decrease of the correlation with increase of distance ρ. The control first brings coherence to the pattern in small local regions in the transverse domain, as manifested in $\zeta(\rho, t = 30)$ by the appearance of a periodic structure in the region of small ρ, but the pattern remains uncorrelated at a long distance, as evidenced by fluctuations of the correlation function around zero for larger values of ρ. The increase in strength of such a periodic structure and its extension to the region of large values of ρ are a quantitative measure of the increase of spatial periodicity of the pattern in the transition from turbulence to a controlled roll state. The correlation function $\zeta(\rho, t = 200)$ for the desired state shows a periodic feature with a spatial period equal to $2\pi/k_x$.

In the time domain the controlled laser signal, both its real and imaginary parts, resembles the main features of a stable roll pattern as described earlier, exhibiting a periodic feature in local regions characterized by two frequencies. Figure 4(a) shows the time evolution of the real part of the laser field with $\omega_{\mathrm{rol}} = 2.9$, as desired, and $\omega_{\mathrm{ref}} = 0.11$, as a modulation corresponding to the reference frequency. The system is stabilized after a transition time of $t \sim 100$ from the onset of the control. The time scale of this temporal transition to the targeted periodic orbit is found to be the same as that of the spatial transition to the transverse periodic pattern, indictive of a simultaneous organization of the system in space and time. We note, however, that the temporal profile and therefore frequency content of the slow time scale modulation, as shown in Fig. 4(a), are slightly altered by the control from that of a roll pattern obtained in the parameter region of stable rolls, in particular, the occurrence of sharp peaks exhibited in this figure. As a result, the controlled laser intensity is modified from the perfect periodic feature of a stable roll. These effects are attributable to the existence of a finite strength of the feedback signal after the system is controlled to the target UPP. Figure 4(b) shows the time evolution of the feedback

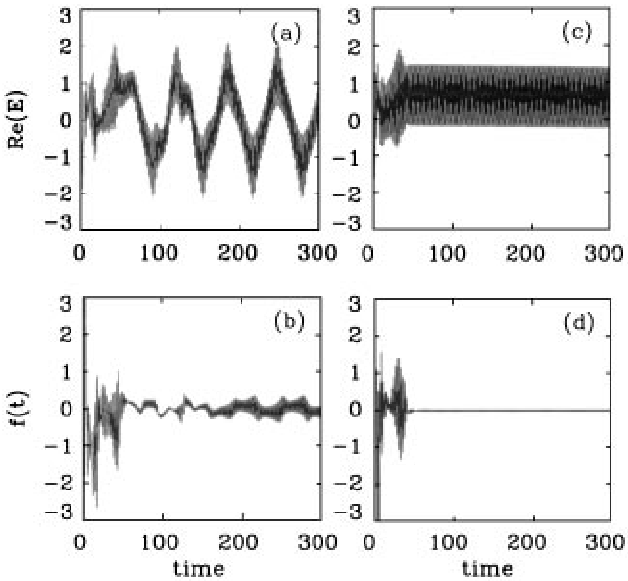

FIG. 4. Conversion of the system, through the control, from chaos to limit cycles in a local region in the transverse domain $c_1 = c_2/2 = 0.46$. (a),(b) The evolution of Re(E) and $f(t)$ for $A = 2.52$, while the other parameters, grid points, and transverse size are the same as those used in Fig. 2. (c),(d) The same time evolution for $A = 2.58$ in the new reference frame ($\omega_{\text{ref}} = 0.22$) and also with 60×60 grid points in the transverse plane of width 17.30. The other parameters are the same as in Fig. 2.

signal recorded in the same local region in the transverse plane as the time series in Fig. 4(a), its strength being reduced to about 15% of Re(E) (peak to peak) after the control has been established. We note that a similar control using laser intensity as an experimentally more accessible control variable has also been demonstrated, which may be readily realized in laser and optical experiments using an array detector interfaced with a computer for the feedback date processing. Moreover, we find that after the UPP is stabilized, the feedback strength from the spatial coupling, which is induced mainly by the limited spatial resolution in the numerical simulation, is weak, the dominant contribution arising from the time-delayed coupling due to the presence of the reference frequency. Earlier investigations of temporal chaos in detuned laser systems have shown that by changing the frequency frame, e.g., by transforming $E \rightarrow E \exp(i\omega_{\text{ref}} t)$ in the equations of motion, the reference frequency can be removed [17]. The control using Re(E) as the control variable in this new reference frame gives rise to sinusoidal-like behavior in the real (and imaginary) part of the laser field as well as the laser intensity. The feedback is consequently reduced significantly through a perfect synchronization of the feedback with the UPP. Figure 4(c) shows control of the system in such a frame where a more appropriate width and grid size in the transverse plane has also been used in minimizing simulation errors induced by limited spatial resolution. The maximum feedback signal as shown in Fig. 4(d) is reduced to $\sim$2% of the real part of the laser output, which arises essentially from the very small difference between the desired period t_0 and the actually controlled period. The stabilization of the UPP in this frequency frame is now maintained in a nearly noninvasive control manner.

The generality of our control algorithm and its robustness to the control parameters has been tested for different turbulent patterns of the system. For given control parameters t_0 and r_0 close to an (intrinsic) unstable roll in the turbulence, e.g., $t_0 \sim t_{\text{rol}} = 2\pi/\omega_{\text{rol}}$ and $r_0 \sim r_{\text{rol}} = 2\pi/k_A$, control is obtained over quite an extensive range of the proportionality coefficient; for instance, $0.8 < c < 3.0$ for $c_1 = c_2 = c$. For c below the critical values for stable rolls, the transverse wave fronts in the directions of $r_\perp$ and $r_\parallel$ cannot be synchronized in one direction, giving rise to coexisting unstable roll structures in both directions in different transverse regions. For c above the critical values, however, both wave fronts can be stabilized in the whole transverse domain through strong feedback coupling, forming a spatially stable rhombic state.

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