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## From Soliton Equations to Integrable Cellular Automata through a Limiting Procedure

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We show a direct connection between a cellular automaton and integrable nonlinear wave equations. We also present the  $N$ -soliton formula for the cellular automaton. Finally, we propose a general method for constructing such integrable cellular automata and their  $N$ -soliton solutions. [S0031-9007(96)00013-0]

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In order to investigate complex physical phenomenon, we have to adopt a simple model which exhibits essential features of the phenomenon. Cellular automata (CA's) serve as such simple models and have been widely investigated in physics, chemistry, biology, and computer sciences [1]. They present a large variety of structures in their evolution. About a decade ago, Park, Steiglitz, and Thurston extracted a notion of *soliton* from them [2]. They found in a filter type of CA that some patterns of nonzero cell values often propagate with fixed finite velocity and they retain their identity after collisions. These behaviors are quite similar to solitary wave solutions (solitons) of nonlinear wave equations such as the Korteweg-de Vries (KdV) equation. Several types of CA's which possess soliton structures have been studied and many features of the systems have been clarified [3-5], though, to our best knowledge, no direct relation of these CA's to the nonlinear wave equations has been reported yet.

Several years ago, two of the authors (D. T. and J. S.) proposed a new type of filter CA [6]. The CA is 1 (space) + 1 (time) dimensional and two valued (0 and 1). The value of the  $j$ th cell at time  $t$ ,  $u_j^t$ , is given as

$$u_j^{t+1} = \begin{cases} 1 & \text{if } u_j^t = 0 \text{ and } \sum_{i=-\infty}^{j-1} u_i^t > \sum_{i=-\infty}^{j-1} u_i^{t+1}, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

where  $u_j^t = 0$  is assumed for  $|j| \gg 1$ . We can put the evolution rule in another way [7]. At time  $t$ , we have an

infinite sequence composed of 0 and 1. The number of 1's is finite. The rule to determine the state at  $t + 1$  is as follows: (1) Move every 1 only once. (2) Exchange the leftmost 1 with its nearest right 0. (3) Exchange the leftmost 1 among the rest of the 1's with its nearest right 0. (4) Repeat this procedure until all of the 1's are moved.

A peculiar feature of the CA is that any state consists only of solitons, interacting in the same manner as KdV solitons (Fig. 1). Moreover, it possesses infinitely many conserved quantities [8]. Hence it was considered to be an analog of integrable nonlinear wave equations. The purpose of the present Letter is to show a direct connection of a class of CA's (or difference-difference equations which take discrete values) with integrable nonlinear wave equations by clarifying the relation of the CA [Eq. (1)] to them. Hence we can present a general method for construction of such *integrable* CA's. Explicit forms of  $N$ -soliton solutions are also presented.

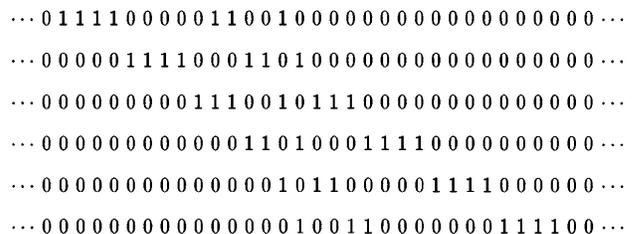


FIG. 1. An example of time evolution of Eq. (1). Three patterns of 1's (1111, 11, and 1) retain their forms with some phase shifts after collisions.

One of the most familiar nonlinear wave equations is the KdV equation

$$\frac{\partial}{\partial t} a(x, t) = \frac{\partial^3}{\partial x^3} a(x, t) + a(x, t) \frac{\partial}{\partial x} a(x, t). \quad (2)$$

The KdV equation is integrable in the sense that it admits Lax representation, has a Hamiltonian structure, and is exactly solvable in some profound nontrivial sense by inverse scattering transform [9].

An integrable discretization (difference-difference equation) of the KdV equation is the Lotka-Volterra equation [10]:

$$\frac{d}{dt} b_j(t) = b_j(t) [b_{j+1}(t) - b_{j-1}(t)]. \quad (3)$$

It appears in a model of struggle for existence of biological species and thin structures of Langmuir oscillations in plasma. By putting  $b_j(t) = 1 + (1/6)\epsilon^2 a((j + 2t)\epsilon, \epsilon^3 t/3)$ , and taking the limit  $\epsilon \rightarrow 0$ , Eq. (3) is transformed into the KdV equation. The Lotka-Volterra equation is also integrable, and hence it has an infinite number of conserved quantities [10,11].

An integrable difference-difference equation related to the Lotka-Volterra equation was proposed by Hirota and Tsujimoto [12]:

$$\frac{c_j^{t+1}}{c_j^t} = \frac{1 + \delta c_{j-1}^t}{1 + \delta c_{j+1}^{t+1}} \quad (j, t \in \mathbf{Z}). \quad (4)$$

This equation also possesses  $N$ -soliton solutions and infinite conserved quantities. One can easily check that the Lotka-Volterra equation is a continuous limit of Eq. (4) by denoting  $c_j^t = b_j(-\delta t)$  and taking  $\delta \rightarrow 0$ . By replacing  $c_j^t$  by  $\exp(d_j^t)$ , we get

$$d_j^{t+1} - d_j^t = \ln \left( \frac{1 + \delta \exp(d_{j-1}^t)}{1 + \delta \exp(d_{j+1}^{t+1})} \right). \quad (5)$$

Now we take an important limiting procedure, which is a key to obtaining integrable CA's. We introduce a positive parameter  $\epsilon \equiv -(\ln \delta)^{-1}$  or  $\delta = e^{-1/\epsilon}$  and set  $d_j^t = e_j^t/\epsilon$ . Then, noticing the fact that

$$F(X) \equiv \lim_{\epsilon \rightarrow +0} \epsilon \ln(1 + e^{X/\epsilon}) = \max[0, X], \quad (6)$$

we obtain from Eq. (5) in the limit  $\epsilon \rightarrow +0$

$$e_j^{t+1} - e_j^t = -F(e_{j+1}^{t+1} - 1) + F(e_{j-1}^t - 1),$$

or, setting  $f_x^{-y+x} \equiv e_y^x$ ,

$$\begin{aligned} f_{j+1}^{t+1} - f_j^t &= -F(f_{j+1}^{t+1} - 1) + F(f_j^{t+1} - 1) \\ &\equiv -(\Delta_j - \Delta_t)F(f_j^t - 1), \end{aligned} \quad (7)$$

where  $\Delta_j X_j^t \equiv X_{j+1}^t - X_j^t$  and  $\Delta_t X_j^t \equiv X_j^{t+1} - X_j^t$ . Note that this equation describes a filter type CA when we restrict values of  $f_j^t$  to integers. (Here we use the term CA in an extended meaning, that is, we allow the dependent variable  $f_j^t$  to take values in all integers.)

We now show that Eqs. (7) and (1) are essentially the same. We can express Eq. (1) as

$$\begin{aligned} u_j^{t+1} &= \min \left( 1 - u_j^t, \sum_{i=-\infty}^{j-1} u_i^t - \sum_{i=-\infty}^{j-1} u_i^{t+1} \right) \\ &= \sum_{i=-\infty}^{j-1} u_i^t - \sum_{i=-\infty}^{j-1} u_i^{t+1} \\ &\quad - \max \left( 0, \sum_{i=-\infty}^{j-1} u_i^t - \sum_{i=-\infty}^{j-1} u_i^{t+1} + u_j^t - 1 \right), \end{aligned} \quad (8)$$

by using min or max function. Introducing  $S_j^t = \sum_{i=-\infty}^j u_i^t$ , we get

$$S_{j+1}^{t+1} - S_j^t = -F(S_{j+1}^t - S_j^{t+1} - 1). \quad (9)$$

This is equivalent to Eq. (7) by setting  $f_j^t = S_{j+1}^t - S_j^{t+1} = (\Delta_j - \Delta_t)S_j^t$ . Thus we have shown the connection of the CA (1) with KdV and Lotka-Volterra equations. The route from the KdV equation to the CA is summarized in Fig. 2.

As we clarified above, Eq. (1) is a (nonanalytic) limit of the difference-difference equation (4) so that we may expect that an  $N$ -soliton solution of Eq. (4) turns into an  $N$ -soliton solution for Eq. (1). This is, indeed, the case. The  $N$ -soliton solution to Eq. (4) is given by the Casorati determinant or the Gram type determinant through Hirota's bilinear identity [13]. By taking the limit  $\epsilon \rightarrow +0$ , we can prove the  $N$ -soliton solution to Eq. (4) really gives the solution to Eq. (1) as

$$u_j^t = \rho_j^t - \rho_j^{t+1} - \rho_{j-1}^t + \rho_{j-1}^{t+1}, \quad (10)$$

with

$$\rho_j^t = \max_{\mu_i=0,1} \left[ \sum_{i=1}^N \mu_i \eta_i - \sum_{i>l} \mu_i \mu_l A_{il} \right], \quad (11)$$

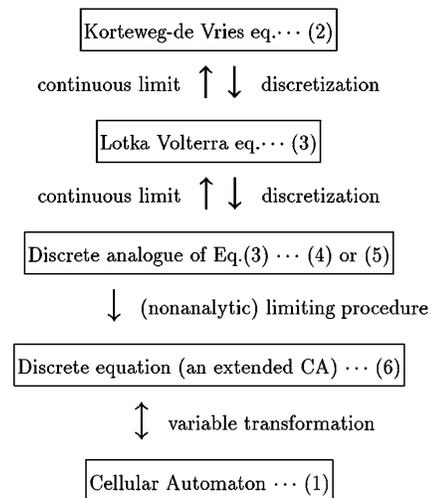


FIG. 2. The route from the KdV equation [Eq. (2)] to the integrable cellular automaton [Eq. (1)]. The numbers on the right side of the boxes correspond to those of equations in the text.

where  $\eta_i = k_{ij} - \omega_i t + \eta_i^0$ ,  $A_{il} = 2 \min(\omega_i, \omega_l)$ , and

$$k_i = \text{sgn}(\omega_i) \min(1, |\omega_i|). \quad (12)$$

Here  $\eta_i^0$  ( $i = 1, 2, \dots, N$ ) are arbitrary integers,  $\omega_i$  ( $i = 1, 2, \dots, N$ ) are arbitrary but either all positive or all negative integers, and  $\max_{\mu_i=0,1}[X(\{\mu_i\})]$  denotes the maximal value in  $2^N$  values of  $X(\{\mu_i\})$  obtained by replacing each  $\mu_i$  by 0 or 1. (We can remove the constraint on  $\omega_i$ , but then  $A_{il}$  has some complicated forms.) For example, a 3-soliton solution is given by

$$\begin{aligned} \rho_j^t = \max[ & 0, \eta_1, \eta_2, \eta_3, \eta_1 + \eta_2 - A_{12}, \\ & \eta_2 + \eta_3 - A_{23}, \eta_3 + \eta_1 - A_{31}, \\ & \eta_1 + \eta_2 + \eta_3 - A_{12} - A_{23} - A_{31}]. \end{aligned}$$

It should be noted that  $\rho_j^t$  satisfies

$$\rho_{j+1}^{t+1} + \rho_j^{t-1} = \max[\rho_{j+1}^t + \rho_j^t, \rho_{j+1}^{t-1} + \rho_j^{t+1} - 1], \quad (13)$$

which is obtained from Eqs. (8) and (10), and may be considered as an analog of bilinear identity.

Once the connection of the CA equation (1) with the integrable nonlinear wave equations is found, it is straightforward to construct other kinds of integrable CA's from other nonlinear integrable equations. We sketch out how to construct integrable CA's in general, from the viewpoint of the so-called Sato theory [14]. In the early 1980s, Sato established a unified theory of solitons. He showed that any integrable differential equation can be regarded as a dynamical system on a universal Grassmann manifold (UGM). A solution to the nonlinear equation corresponds to a point of UGM. It is called the  $\tau$  function. Using the Plücker relation of UGM, we obtain Hirota's bilinear identity for the  $\tau$  function. Date, Jimbo, Kashiwara, and Miwa developed the Sato theory giving its link with infinite dimensional Lie algebras by the method of field theory and vertex operators [15]. Then the  $\tau$  function is expressed as a vacuum expectation value of a fermion field operator.

In terms of usual fermion creation and annihilation operators, which satisfy  $[\psi_i, \psi_j]_+ = [\psi_i^*, \psi_j^*]_+ = 0$ , and  $[\psi_i, \psi_j^*]_+ = \delta_{i,-j}$  ( $i, j \in \mathbf{Z} + 1/2$ ), the  $N$ -soliton solution is expressed as [16]

$$\tau(\mathbf{t}) = \langle \text{vac} | \prod_{i=1}^N [1 + c_i \psi(p_i, \mathbf{t}) \psi^*(q_i, \mathbf{t})] | \text{vac} \rangle, \quad (14)$$

where  $\psi(p, \mathbf{t}) \equiv e^{\xi(\mathbf{t}, p)} \psi(p)$ ,  $\psi^*(q, \mathbf{t}) \equiv e^{-\xi(\mathbf{t}, q)} \psi^*(q)$ ,  $\psi(p) \equiv \sum_{j \in \mathbf{Z} + 1/2} \psi_j p^{-j-1/2}$ , and  $\psi^*(q) \equiv \sum_{j \in \mathbf{Z} + 1/2} \psi_j^* q^{-j-1/2}$ . Here  $c_i$  is an arbitrary constant,  $\mathbf{t}$  denotes the time variables, and a function  $\xi(\mathbf{t}, p)$  is arbitrary in principle, though we need a careful choice in order to get a significant differential or difference equation.

In order to construct integrable CA's, we should put  $\mathbf{t} = \{t, j_1, j_2, \dots, j_m\}$  and impose the condition:

$e^{\xi(\mathbf{t}, p) - \xi(\mathbf{t}, q)} = e^{(-\omega t + k_1 j_1 + \dots + k_m j_m)/\epsilon}$ , where two of the  $\omega$  and  $k_j$ 's are arbitrary integers. This condition gives the dispersion relation  $\omega = \omega(k_1, \dots, k_m; \epsilon)$  for we have only two free parameters  $p$  and  $q$ . At the same time,  $p$  and  $q$  have asymptotic forms:  $p = e^{P/\epsilon} + \dots$  and  $q = e^{Q/\epsilon} + \dots$ . Thus putting  $\rho(\mathbf{t}) \equiv \lim_{\epsilon \rightarrow +0} \epsilon \ln[\tau(\mathbf{t})] \epsilon \rightarrow +0$  with some careful choice of coefficients of the bilinear identity  $\rho(\mathbf{t})$  satisfies an equation similar to Eq. (13), from which we obtain an  $m$ -dimensional CA and its  $N$ -soliton solutions. Since the CA is thus constructed, it naturally inherits the geometrical and algebraic nature of the  $\tau$  function.

To be more specific, let  $\mathbf{t} = \{t, j\}$ ,  $e^{\xi(\mathbf{t}, p)} = p^j (p + \delta)^{-j} (p + 1 + \delta)^{j-t}$ , and  $q = -p - 2\delta - 1$ . The general consideration given in Ref. [16] leads to the bilinear identity for  $\tau_j^t \equiv \tau(\mathbf{t})$ :

$$\tau_{j+1}^t \tau_j^t - (1 + \delta) \tau_j^{t-1} \tau_{j+1}^{t+1} + \delta \tau_j^{t+1} \tau_{j+1}^{t-1} = 0. \quad (15)$$

The condition  $e^{\xi(\mathbf{t}, p) - \xi(\mathbf{t}, q)} = e^{(-\omega t + k_j)/\epsilon}$  implies  $p = -\delta - (1 + e^{-\omega/\epsilon})^{-1}$  and

$$k = -\epsilon \ln \left[ \frac{e^{-\omega/\epsilon} + \delta(1 + e^{-\omega/\epsilon})}{1 + \delta(1 + e^{-\omega/\epsilon})} \right]. \quad (16)$$

Then we can show that  $\tau_j^t$  is always positive for any integers  $\omega_i$ 's and suitable  $c_i$ 's, so that, taking  $\delta = e^{-1/\epsilon}$ , Eq. (15) is rewritten as

$$\begin{aligned} \epsilon \ln[(1 + e^{-1/\epsilon}) \tau_j^{t-1} \tau_{j+1}^{t+1}] = \epsilon \ln[ & \tau_{j+1}^t \tau_j^t \\ & + e^{-1/\epsilon} \tau_j^{t+1} \tau_{j+1}^{t-1}]. \end{aligned} \quad (17)$$

Thus, denoting  $\rho_j^t = \lim_{\epsilon \rightarrow +0} \epsilon \ln \tau_j^t$ , we find from Eq. (17) that  $\rho_j^t$  satisfies Eq. (13) and (16) reduces to (12). Since the right side of Eq. (14) is calculated as  $\det \mathbf{G}(\mathbf{t})$ , where the  $i, j$  element of the  $N \times N$  matrix  $\mathbf{G}(\mathbf{t})$  is given as

$$[\mathbf{G}(\mathbf{t})]_{i,j} = \delta_{i,j} + \frac{c_i}{p_i - q_j} e^{\xi(\mathbf{t}, p_i) - \xi(\mathbf{t}, q_j)},$$

it is also easily seen that the  $N$ -soliton solution (14) turns into (11).

We can construct other types of integrable CA's with this method, for example the CA's related to the generalized Lotka-Volterra systems, one- and two-dimensional Toda lattices, and the Kadomtsev-Petviashvili equations. We shall report them with detailed analysis of the solutions and conserved quantities in separate papers.

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