Test of the Running of α_s in τ Decays

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The τ decay rate into hadrons of invariant mass smaller than $\sqrt{s_0} \gg \Lambda_{\text{QCD}}$ can be calculated in QCD assuming global quark-hadron duality. It is shown that this assumption holds for $s_0 > 0.7$ GeV^2. From measurements of the hadronic mass distribution, the running coupling constant $\alpha_s(s_0)$ is extracted in the range 0.7 GeV$^2 < s_0 < m_\tau^2$. At $s_0 = m_\tau^2$, the result is $\alpha_s(m_\tau^2) = 0.329 \pm 0.030$. The running of α_s is in good agreement with the QCD prediction.

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The scale dependence of coupling constants is one of the key features of renormalizable quantum field theories. In QCD, the effective coupling constant $\alpha_s(Q^2)$ is predicted to decrease with the momentum transfer Q^2, a property referred to as asymptotic freedom [1]. This prediction has been tested by comparing data obtained from experiments operating at different energies [2]; it has also been studied in single high-energy experiments at ep and $p\overline{p}$ colliders, where a large range in Q^2 can be probed simultaneously [3]. Here we propose a test of the scale dependence of $\alpha_s(Q^2)$ in the low-energy region 0.7 GeV$^2 < Q^2 < m_\tau^2$. Our method is based on integrals of the invariant mass distribution in hadronic τ decays. It provides a unique opportunity to test one of the most important predictions of QCD in a single experiment and at low energies, where the effect of the running of α_s is most pronounced.

We shall consider the τ decay rate into hadrons of invariant mass squared smaller than s_0, normalized to the leptonic decay rate,

$$R_\tau(s_0) = \frac{\Gamma(\tau \to \nu_\tau + \text{hadrons}; s_{\text{had}} < s_0)}{\Gamma(\tau \to \nu_\tau e \overline{\nu}_e)}$$
$$= \int_0^{s_0} ds \, \frac{dR_\tau(s)}{ds}, \qquad (1)$$

where dR_τ/ds is the inclusive hadronic spectrum. As long as $s_0 \gg \Lambda_{\text{QCD}}^2$, the quantity $R_\tau(s_0)$ can be calculated in QCD using the operator product expansion (OPE) [4,5]. Applying the OPE in the physical region assumes global quark-hadron duality, i.e., that decay rates admit a QCD description after a "smearing" over a sufficiently wide energy interval has been performed [6], which in the present case is provided by the integration over the range $0 < s < s_0$. The question of how accurate this assumption is and for what values of s_0 it applies is a phenomenological one; it cannot be answered yet from theoretical grounds. Below, we shall investigate this question, comparing data with theoretical predictions based on the duality assumption. A similar test of duality

has been performed in Ref. [7], using data on the $e^+e^- \to$ hadrons cross section.

The τ decay rate into hadrons can be written in terms of moments $\mathcal{M}_k^{(J)}$ of the absorptive part of current-current correlation functions of angular momentum J [8,9]. The quantity $R_\tau(s_0)$ is given by

$$\frac{1}{3S_{\text{EW}}} R_\tau(s_0) = \frac{2s_0}{m_\tau^2} \mathcal{M}_0^{(1)}(s_0) - 2\left(\frac{s_0}{m_\tau^2}\right)^3 \mathcal{M}_2^{(1)}(s_0)$$
$$+ \left(\frac{s_0}{m_\tau^2}\right)^4 \mathcal{M}_3^{(1)}(s_0) + \frac{2s_0}{m_\tau^2} \mathcal{M}_0^{(0)}(s_0)$$
$$- 2\left(\frac{s_0}{m_\tau^2}\right)^2 \mathcal{M}_1^{(0)}(s_0)$$
$$+ \frac{2}{3}\left(\frac{s_0}{m_\tau^2}\right)^3 \mathcal{M}_2^{(0)}(s_0), \qquad (2)$$

where $S_{\text{EW}} \simeq 1.0194$ accounts for electroweak radiative corrections [10]. The moments can be written as contour integrals along a circle of radius s_0 in the complex plane. Since the only large mass scale in these integrals is s_0, the OPE provides an expansion in powers of $1/s_0$,

$$\mathcal{M}_k^{(J)}(s_0) = \mathcal{M}_k^{(1)}[\alpha_s(s_0)]_{\text{pert}} \delta_{J=1}$$
$$+ \sum_{n=1}^{\infty} c_n^{(J)}[\alpha_s(s_0)] \frac{\langle O_{2n}\rangle}{s_0^n}. \qquad (3)$$

The leading term is given by perturbation theory alone. Terms suppressed by powers of $1/s_0$ consist of perturbative coefficients $c_n^{(J)}$ multiplying dimensionful parameters $\langle O_{2n}\rangle$, such as quark masses or vacuum condensates [4]. This is how nonperturbative effects are incorporated. There is no leading term for the moments with $J = 0$, which vanish in the chiral limit and are thus proportional to powers of the light quark masses. For the moments with $J = 1$, the perturbative contribution is

$$\mathcal{M}_k^{(1)}[\alpha_s(s_0)]_{\text{pert}} = 1 + \sum_{n=1}^{\infty} d_n^{(k)}\left(\frac{\alpha_s(s_0)}{\pi}\right)^n, \quad (4)$$

where $\alpha_s(s_0)$ is defined in the minimal subtraction ($\overline{MS}$) renormalization scheme, $d_1^{(k)} = 1$, and the next three coefficients are given by [8,9]

$$d_2^{(k)} = 1.639\,82 + \frac{9}{4(k+1)},$$

$$d_3^{(k)} = -10.2839 + \frac{11.3792}{k+1} + \frac{81}{8(k+1)^2},$$

$$d_4^{(k)} = K_4 - 155.955 - \frac{46.238}{k+1} + \frac{94.810}{(k+1)^2}$$
$$+ \frac{68.344}{(k+1)^3}. \tag{5}$$

The coefficient K_4 appears in the perturbative expansion of the Adler function and is not known exactly. An estimate using the principle of minimal sensitivity and the effective charge approach [11] gives $K_4 \simeq 27.5$ [12]. We

shall use this result in our analysis. The error due to the truncation of the perturbation series in (4) is of the order of the last term included. It can also be estimated by summing a subset of corrections to all orders in perturbation theory. Such a class of corrections is provided by the renormalon chains [13], which are the terms of order $\beta_0^{n-1}\alpha_s^n$, where β_0 is the first coefficient of the β function. For the case of the moments, the resummation of these terms has been discussed in Refs. [9,14]. Below, we shall take fixed-order perturbation theory as the nominal scheme and use the resummation of renormalon chains to estimate the perturbative uncertainty.

The nonperturbative corrections in the OPE are proportional to the light quark masses or to vacuum condensates [4]. We quote the power corrections for the sum of the moments contributing to $R_\tau(s_0)$ in (2). The terms relevant to the numerical analysis are

$$\frac{1}{3S_{EW}} R_\tau(s_0)|_{power} = -6|V_{us}|^2 \frac{m_s^2(s_0)}{m_\tau^2}\left[1 + \frac{s_0}{m_\tau^2} - \left(\frac{s_0}{m_\tau^2}\right)^2 + \frac{1}{3}\left(\frac{s_0}{m_\tau^2}\right)^3\right] + \frac{16\pi^2}{m_\tau^4}[\langle m_u\overline{\psi}_u\psi_u\rangle + |V_{ud}|^2\langle m_d\overline{\psi}_d\psi_d\rangle$$
$$+ |V_{us}|^2\langle m_s\overline{\psi}_s\psi_s\rangle] - \frac{512\pi^3}{27}\frac{\rho\alpha_s\langle\overline{\psi}\psi\rangle^2}{m_\tau^6} + \cdots, \tag{6}$$

where $m_s(s_0)$ is the running strange-quark mass, $\langle m_q\overline{\psi}_q\psi_q\rangle$ are the quark condensates, and $\rho\alpha_s\langle\overline{\psi}\psi\rangle^2$ denotes the four-quark condensate. More detailed expressions, which are used in our analysis, can be found elsewhere [5,8,9]. At tree level, the powers of $1/s_0$ appearing in the OPE of the moments in (3) conspire with the powers of s_0/m_τ^2, which multiply the moments in (2), so that the nonperturbative corrections to $R_\tau(s_0)$ are suppressed by powers of $1/m_\tau^2$. This is no longer the case if radiative corrections to the coefficients of the vacuum condensates are taken into account, but the corresponding effects are very small. As a consequence, the power corrections to $R_\tau(s_0)$ remain small down to rather low values of s_0; using standard values of the QCD parameters (which we take from Ref. [9]) we find $-(1.4 \pm 0.5)\%$ for the right-hand side of (6) at $s_0 = m_\tau^2$, and $-(1.5 \pm 0.5)\%$ at $s_0 = 1\,\text{GeV}^2$. This observation, together with the fact that the perturbative contributions are known to high order, guarantees a good convergence of the OPE down to low-energy scales.

To extract the quantity $R_\tau(s_0)$, we use the spectra of the hadronic mass distribution reported by the CLEO and ALEPH collaborations [15,16] (see Fig. 1). To obtain dR_τ/ds, we multiply the normalized distributions by the world average $R_\tau = R_\tau(m_\tau^2) = 3.642 \pm 0.010$ [17]. Not shown is the contribution from $\tau \rightarrow h^-\nu_\tau$ with $h^- = \pi^-$ or K^-, which has a branching ratio of $(11.77 \pm 0.14)\%$ [18]. We integrate these spectra over s, combine the results weighted by their statistical errors, and add the systematic errors, which we estimate by taking the difference

between the CLEO and ALEPH data. [As the ALEPH data are preliminary, this estimate may be taken with caution. However, since inclusive quantities such as $R_\tau(s_0)$ do only probe gross features of the hadronic mass distribution, systematic errors play a minor role in our analysis.] This is justified, since the dominant sources of systematic errors are different in the two analyses. The result is shown in Fig. 1. It is represented as a band, since the errors in the $R_\tau(s_0)$ values are strongly correlated. The two
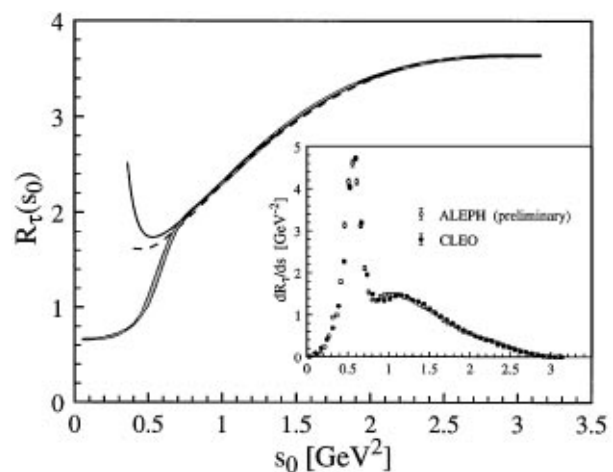

FIG. 1. The quantity $R_\tau(s_0)$ extracted from the data on the hadronic mass distribution dR_τ/ds reported by the CLEO and ALEPH collaborations [15,16] (inset). The experimental result is represented as a band. The curves show the theoretical predictions (see text).

curves show theoretical calculations of $R_\tau(s_0)$ based on the OPE approach outlined above. The solid line is obtained using fixed-order perturbation theory to order α_s^4. The dashed line is obtained by adding to this a resummation of renormalon chains of order α_s^5 and higher, using the results of Ref. [9]. The value of $\alpha_s(m_\tau^2)$ has been adjusted so as to fit the data at $s_0 = m_\tau^2$. The central values obtained in the two schemes are $\alpha_s(m_\tau^2) = 0.329$ (fixed order) and $\alpha_s(m_\tau^2) = 0.309$ (resummed). Their difference provides an estimate of the uncertainty due to unknown higher-order corrections, which is more conservative than that obtained by omitting the term of order α_s^4 in the fixed-order calculation. Varying the values of the nonperturbative parameters within conservative limits changes $\alpha_s(m_\tau^2)$ by up to 2%. Adding linearly the perturbative uncertainty (± 0.020), the nonperturbative uncertainty (± 0.006), and the experimental uncertainty (± 0.004), we find

$$\alpha_s(m_\tau^2) = 0.329 \pm 0.030,$$
$$\alpha_s(m_Z^2) = 0.119 \pm 0.004. \tag{7}$$

For the sake of completeness, we have translated our result into a value of α_s at the mass of the Z boson.

The assumption of global quark-hadron duality can be tested by comparing the data for the quantity $R_\tau(s_0)$ at values $s_0 < m_\tau^2$ with the theoretical prediction [8]. Given $\alpha_s(m_\tau^2)$, the value of $\alpha_s(s_0)$ follows from the solution of the renormalization-group equation

$$\mu^2 \frac{d\alpha_s(\mu^2)}{d\mu^2} = -\alpha_s(\mu^2)\beta[\alpha_s(\mu^2)],$$

$$\beta(\alpha_s) = \beta_0 \frac{\alpha_s}{4\pi} + \beta_1 \left(\frac{\alpha_s}{4\pi}\right)^2 + \beta_2 \left(\frac{\alpha_s}{4\pi}\right)^3 + \cdots, \tag{8}$$

where $\beta_0 = 9$, $\beta_1 = 64$, and $\beta_2 = 3863/6$ are the first three coefficients of the β function, evaluated for $n_f = 3$ light quark flavors. (The value of β_2 is specific to the $\overline{MS}$ scheme.) Whereas the theoretical uncertainties are a limiting factor in the determination of $\alpha_s(m_\tau^2)$, they have little influence on the s_0 dependence of $R_\tau(s_0)$. For the perturbative part of the calculation this is apparent from the good agreement of the two curves in Fig. 1, which refer to values of $\alpha_s(m_\tau^2)$ that differ by 9%. Varying the values of the nonperturbative parameters has a negligible effect ($\sim 0.5\%$ at $s_0 = 1$ GeV^2) on the value of $R_\tau(s_0)$ for $s_0 < m_\tau^2$, since the only dependence on s_0 in (6) comes from the quark-mass corrections, which are known with higher accuracy than the vacuum condensates. Hence, the s_0 dependence of $R_\tau(s_0)$ is predicted essentially without free parameters, and the comparison between theory and experiment provides a direct test of quark-hadron duality. We find good agreement over the entire range 0.7 GeV$^2 < s_0 < m_\tau^2$, indicating that in τ decays duality holds as soon as the integral over the hadronic mass distribution includes the ρ resonance peak. The onset of duality happens almost instantaneously, in accordance with general expectations.

From now on, we shall rely on this behavior and assume that for $s_0 > 0.7$ GeV^2 possible violations of duality can be neglected. We then turn to the main focus of our study: a test, at low energies, of the QCD prediction (8) for the running of the coupling constant. τ decays are an ideal place to study this phenomenon since the value of $\alpha_s(s_0)$ changes by a factor of 2 over the region where duality holds, which is about the same change as in the region between 5 and 100 GeV. From the quantity $R_\tau(s_0)$ shown in Fig. 1, we extract $\alpha_s(s_0)$ as a function of s_0 by fitting to the data the theoretical prediction obtained from (2) and (4)–(6). The result, including experimental errors only, is represented by the dark band in Fig. 2. Theoretical uncertainties arise from the truncation of the perturbation series and from the uncertainty in the values of the nonperturbative parameters. As discussed above, they affect the overall scale of the α_s values [by about (8–10)%], but have very little effect on the evolution of the coupling constant. The sum of the experimental and theoretical errors is represented by the light band. The dashed curve shows the QCD predictions for $\alpha_s(s_0)$ obtained at three-loop order, normalized to the central value of the data at $s_0 = m_\tau^2$. The observed scale dependence of the running coupling constant is in good agreement with the QCD prediction. The small oscillation of the experimental band around the theoretical curve, which could be due to some deviation from duality in the a_1 region, are not significant given the precision of the data.

To quantify this agreement, we extract from the data the β function that describes according to (8) the running of $\alpha_s(s_0)$. Defining $x = \alpha_s(s_0)/4\pi$, we have

$$-\frac{4\pi}{\alpha_s^2(s_0)} \frac{d\alpha_s(s_0)}{d\ln s_0} = \frac{\beta(x)}{x} = \beta_0 + \beta_1 x + \beta_2 x^2 + \cdots. \tag{9}$$
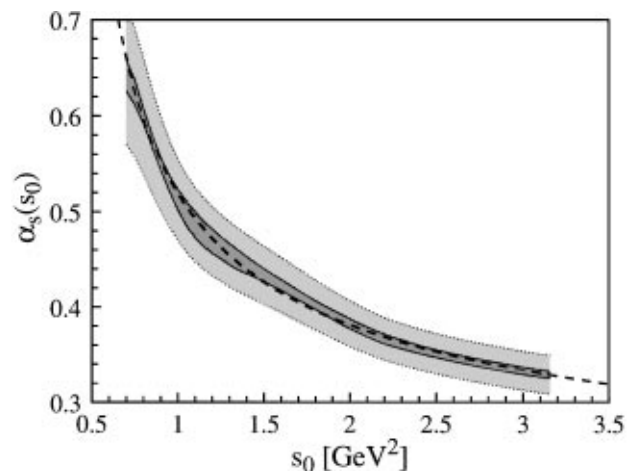

FIG. 2. Values of $\alpha_s(s_0)$ extracted from the data on $R_\tau(s_0)$. The dark band represents the experimental errors, the light one the sum of the experimental and theoretical errors. The errors are strongly correlated. The dashed line shows the three-loop QCD prediction for the running coupling constant.

We approximate the derivative $d\alpha_s/d\ln s_0$ by a ratio of differences, $\Delta\alpha_s/\Delta\ln s_0$, for a set of s_0 values chosen such that the differences $\Delta\alpha_s$, are large enough to be significant given the errors in the measurement. For $\alpha_s(s_0)$ in (9) we take the central value of each interval. We use the following s_0 values: 0.75, 0.95, 1.35, 2.06, and 3.16 GeV^2, corresponding to four intervals of increasing width $\Delta\ln s_0$, but constant $\Delta\alpha_s \simeq 0.075$. The results are shown in Fig. 3. The estimate of the errors includes the theoretical uncertainties, the error due to the choice of finite intervals in α_s, and the experimental errors, which in this case are the dominant ones. The curves in Fig. 3 show the QCD β function at one-, two-, and three-loop order in perturbation theory. The data provide clear evidence for the running of the coupling constant. Moreover, they prefer a running that is stronger than predicted at one-loop order. Between the three curves, the one that shows the three-loop prediction provides the best description of the data. Performing a fit with the three-loop β function, where $\beta_0 = 9$ and $\beta_1 = 64$ are kept fixed but the three-loop coefficient β_2 is treated as a parameter, we find $\beta_2^{\text{exp}}/\beta_2^{\text{th}} = 1.6 \pm 0.7$. We believe that such an experimental determination of the β function beyond the leading order can at present be done only in τ decays. [A high-precision measurement of $R_{e^+e^-}(s)$ in the region below the charmonium resonance would provide an alternative place for such a study.] At higher energies, the value of α_s is too small to distinguish between the three curves in Fig. 3; measurements in the region $Q \sim 100$ GeV, for instance, correspond to values $x \sim 0.01$.

In summary, we have presented a method to measure the running coupling constant $\alpha_s(s)$ in the low-energy region 0.7 GeV$^2 < s < m_\tau^2$ using τ decay data obtained in a single experiment. It provides a test of one of the key features of QCD in a region where the effect of the running of α_s is most pronounced. The theoretical analysis is based on the OPE and the assumption of global quark-hadron duality. We have tested this assumption and find that it seems to hold if the τ decay rate is integrated over an energy interval large enough to include the ρ resonance peak. Our analysis provides a test of QCD at scales comparable with the lowest ones attainable before ($Q^2 \simeq 2.5$ GeV^2 in deep-inelastic scattering), and with higher precision than all other single measurements of the running to date. We have extracted for the first time the β function from data and find that it is in good agreement with the three-loop prediction of QCD.
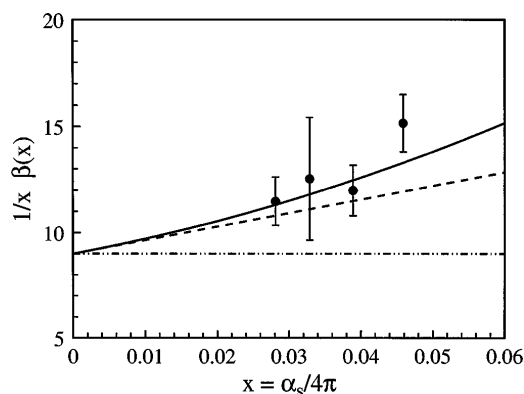
FIG. 3. Experimental determination of the β function. The curves show the QCD prediction at one-loop (dash-dotted), two-loop (dashed), and three-loop (solid) order.

[1] D. J. Gross and F. Wilczek, Phys. Rev. Lett. **30**, 1343 (1973); H. D. Politzer, *ibid.* **30**, 1346 (1973).

[2] B. Webber, in *Proceedings of the 27th International Conference on High Energy Physics, Glasgow, Scotland, 1994,* edited by P. J. Bussey and I. G. Knowles (IOP Publishing, Bristol, 1995), Vol. 1, p. 213.

[3] ZEUS Collaboration, M. Derrick *et al.,* Phys. Lett. B **363**, 201 (1995); W. T. Giele, E. W. N. Glover, and J. Yu, Report No. FERMILAB-Pub-95/127–T, 1995 [hep-ph/9506442].

[4] M. A. Shifman, A. I. Vainstein, and V. I. Zakharov, Nucl. Phys. **B147**, 385 (1979); **B147**, 448 (1979).

[5] E. Braaten, S. Narison, and A. Pich, Nucl. Phys. **B373**, 581 (1992).

[6] E. C. Poggio, H. R. Quinn, and S. Weinberg, Phys. Rev. D **13**, 1958 (1976).

[7] S. I. Eidelman, L. M. Kurdadze, and A. I. Vainshtein, Phys. Lett. **82B**, 278 (1979).

[8] F. Le Diberder and A. Pich, Phys. Lett. B **289**, 165 (1992).

[9] M. Neubert, Nucl. Phys. B **463**, 511 (1996).

[10] W. J. Marciano and A. Sirlin, Phys. Rev. Lett. **56**, 22 (1986).

[11] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981); G. Grunberg, Phys. Lett. B **221**, 70 (1980); Phys. Rev. D **29**, 2315 (1984).

[12] A. L. Kataev and V. V. Starshenko, Mod. Phys. Lett. A **10**, 235 (1995).

[13] G. 't Hooft, in *Proceedings of the 15th International School of Subnuclear Physics, Erice, Sicily, 1977,* edited by A. Zichichi (Plenum Press, New York, 1979), p. 943.

[14] P. Ball, M. Beneke, and V. M. Braun, Nucl. Phys. **B452**, 563 (1995).

[15] CLEO Collaboration, T. Coan *et al.,* Phys. Lett. B **356**, 580 (1995).

[16] L. Duflot, Nucl. Phys. B (Proc. Suppl.) **40**, 37 (1995).

[17] L. Montanant *et al.,* Phys. Rev. D **50**, 1173 (1994); 1995 off-year partial update (URL: http://pdg.lbl.gov/).

[18] R. Stroynowski, Nucl. Phys. B (Proc. Suppl.) **40**, 569 (1995).