Stability and the Fractal Structure of a Spherical Flame in a Self-Similar Regime

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Stability of a discontinuous flame front propagating in a self-similar regime $R = At^{\alpha}$ is considered. It is shown that the regime with $\alpha > 1$ is more unstable compared to the flame propagating with a constant front velocity. The destabilizing effect is more pronounced for flames with the ratio of the fuel density to the density of the products of burning $\rho_1/\rho_2 > 2$ which is typical for usual laboratory flames. A formula for the velocity of a flame with a fractal structure of the front is discussed.

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It is well known that a flame front propagating in a premixed gaseous fuel is hydrodynamically unstable against Landau-Darrieus instability [1–3]. Recent experiments on spherical flames [4] showed that the flame instability leads to the self-similar regime of the front propagation. In the self-similar regime the radius of the flame front changes with time as

$$R = At^{\alpha}, \qquad \dot{R} = \alpha A^{1/\alpha} R^{(\alpha-1)/\alpha}, \tag{1}$$

where A is a coefficient which presumably depends upon the flame parameters. The approximate value of the exponent α measured experimentally for several mixtures is $\alpha \approx 3/2$. The self-similar regime is associated with the development of a fractal structure on the flame front with total surface of the front,

$$S = 4\pi R^2 \frac{\alpha A^{1/\alpha}}{\Theta u_f} R^{(\alpha-1)/\alpha}, \tag{2}$$

where u_f is the flame velocity and Θ is the ratio of the fuel density ρ_1 to the density of the products of burning ρ_2. According to Eq. (2) the fractal dimension of the unstable flame front is $2 + (\alpha - 1)/\alpha$, so that the fractal excess is $d = (\alpha - 1)/\alpha$. Qualitatively the same behavior of a spherical flame was observed in numerical simulations of model nonlinear equations for a flame front [5,6], though different values for the fractal dimension were obtained in different papers.

The fractal structure of a flame front may be described as cascading cells: cells of smaller scale are imposed on cells of large scale and so on. The total surface of a fractal flame may be estimated in the following simple way [4]. Let every step of the cascade decrease the cell size by factor b, $L_{k+1} = L_k/b$, and increase the front surface by factor β, $S_{k+1} = \beta S_k$. The cascading process is limited from below, since the cell size cannot be less than the cutoff wavelength λ_c, for which thermal conduction suppresses the instability [7–11]. The cell size is limited from above too since perturbations with a length scale larger than $2\pi R/n_c$ are stable due to the spherical geometry [12,13] (n_c is the critical number of a spherical function). The fractal structure implies a large number of cascades $N = \ln(2\pi R/\lambda_c n_c)/\ln(b) \gg 1$ and the total flame surface is

$$S_f = S_N = 4\pi R^2 \beta^N = 4\pi R^2 \left(\frac{2\pi R}{n_c \lambda_c}\right)^d, \tag{3}$$

where the fractal excess $d = \ln(\beta)/\ln(b)$. In Eq. (3) the cutoff wavelength $\lambda_c = \lambda_c(\Theta)$ is known from the linear stability theory of a planar flame front [7–11], and b and β should be determined from the nonlinear theory of a curved stationary flame. To obtain the critical number n_c one must consider stability of a spherical flame taking into account the self-similar regime of flame propagation, Eq. (1).

In [4] it was assumed that the flame acceleration in the self-similar regime influences the cutoff wavelength too, which was taken in the form $\lambda_c \sim \sqrt{\chi u_f/\ddot{R}}$. This assumption may be true if $\alpha > 2$, but it is definitely erroneous for the experimentally observed self-similar regimes with $\alpha < 2$. Indeed, in this case the acceleration of the front $\ddot{R} = \alpha(\alpha - 1)t^{\alpha-2}$ tends to zero for $t \to \infty$ and cannot be the main reason of the flame front instability.

In the present Letter we consider stability of a discontinuous flame front propagating in the self-similar regime, Eq. (1). We show that the regime with $\alpha > 1$ is more unstable compared to the flame propagating with a constant front velocity. The destabilizing effect is more pronounced for flames with the expansion coefficient $\Theta = \rho_1/\rho_2 > 2$, which is typical for usual laboratory flames. The obtained results specify the parameter A in Eq. (1) as a function of the expansion coefficient Θ and the exponent α of the self-similar law.

Since the flow is incompressible, the hydrodynamic equations describing flame propagation and stability can

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be taken in the form

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u) + \nabla_\perp \mathbf{w} = 0\,, \tag{4}$$

$$\frac{\partial \mathbf{w}}{\partial t} + u\frac{\partial \mathbf{w}}{\partial r} + (\mathbf{w}\nabla_\perp)u = -\frac{1}{\rho}\nabla_\perp P\,, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + (\mathbf{w}\nabla_\perp)u = -\frac{1}{\rho}\frac{\partial P}{\partial r}\,, \tag{6}$$

where $u, \mathbf{w}$ are the radial and the angular velocity components, respectively, and $\nabla_\perp$ is the angular part of the operator ∇. For the unperturbed spherical flame we have $\mathbf{w} = 0$ and the flow ahead of the flame front is given by

$$u = B(t)/r^2, \quad (P - P_\infty)/\rho_1 = \dot{B}/r - B^2/2r^4. \tag{7}$$

The gas behind the flame front is at rest: $u = 0$ for $r < R$. The boundary conditions at the spherical flame front

$$\rho_1(u_1 - \dot{R}) = -\rho_2\dot{R}\,, \tag{8}$$

$$P_1 + \rho_1(u_1 - \dot{R})^2 = P_2 + \rho_2\dot{R}^2 \tag{9}$$

give the relation $B(t) = R^2\dot{R}(\Theta - 1)/\Theta$. Then the unperturbed flow ahead of the flame front ($r > R$) is described by

$$u = \frac{\Theta - 1}{\Theta}\frac{R^2}{r^2}\dot{R}\,, \tag{10}$$

$$P/\rho_1\dot{R}^2 = \frac{\Theta - 1}{\Theta}\frac{3\alpha - 1}{\alpha}\frac{R}{r} - \frac{1}{2}\frac{(\Theta - 1)^2}{\Theta^2}\frac{R^4}{r^4}\,. \tag{11}$$

Pressure behind the flame front ($r < R$) follows from Eqs. (9)–(11),

$$P/\rho_1\dot{R}^2 = \frac{\Theta - 1}{\Theta}\frac{3\alpha - 1}{\alpha} + \frac{1}{2}\frac{\Theta^2 - 1}{\Theta^2}\,. \tag{12}$$

The linearized hydrodynamic equations for small perturbations of the spherical self-similar flame have the form

$$r^{-1}\frac{\partial}{\partial r}(r^2\tilde{u}) + \hat{l}\tilde{\mathbf{w}} = 0\,, \tag{13}$$

$$\frac{\partial}{\partial t}\hat{l}\tilde{\mathbf{w}} + u\frac{\partial}{\partial r}\hat{l}\tilde{\mathbf{w}} = -\frac{1}{\rho r}\hat{l}^2\tilde{P}\,, \tag{14}$$

$$\frac{\partial \tilde{u}}{\partial t} + \frac{\partial}{\partial r}(u\tilde{u}) = -\frac{1}{\rho}\frac{\partial \tilde{P}}{\partial r}\,, \tag{15}$$

where perturbations are denoted by a tilde and the operator $\hat{l} = r\nabla_\perp$ is introduced. The perturbed boundary conditions at the flame front are

$$\tilde{u}_1 - \partial\tilde{R}/\partial t + (\partial u/\partial r)_1\tilde{R} = \Theta^{-1}(\tilde{u}_2 - \partial\tilde{R}/\partial t)\,, \tag{16}$$

$$\tilde{P}_1 + (\partial P/\partial r)_1\tilde{R} = \tilde{P}_2\,, \tag{17}$$

$$\hat{l}\tilde{\mathbf{w}}_1 + \frac{u_1}{R}\hat{l}^2\tilde{R} = \hat{l}\tilde{\mathbf{w}}_2\,. \tag{18}$$

As an additional boundary condition we take the Landau condition that the velocity of flame propagation is not changed by the small perturbations, i.e.,

$$\tilde{u}_1 - \partial\tilde{R}/\partial t + (\partial u/\partial r)_1\tilde{R} = 0\,. \tag{19}$$

It should be mentioned that the condition Eq. (19) applied to the self-similar regime Eq. (1) does not mean a constant velocity of the flame front any more, as it was in the original statement by Landau and Darrieus [1–3]. Instead it means that the instanteneous normal velocity $\dot{R}$ remains unchanged by perturbations of an infinitely thin front in the linear regime.

Equations (14)–(20) may be solved in a way similar to the classical work by Istratov and Librovich [12]. The flow ahead of the flame front is potential $\tilde{u} = \partial\tilde{\varphi}/\partial r$, $\tilde{\mathbf{w}} = \nabla_\perp\tilde{\varphi}$, $\Delta\tilde{\varphi} = 0$. Taking into account the spherical symmetry of the unperturbed flow, we obtain perturbations ahead of the flame front in the form

$$\tilde{\varphi} = \dot{R}R\Phi_1(t)Y_{n,m}(R/r)^{n+1}, \tag{20}$$

$$\tilde{u} = -(n + 1)\dot{R}\Phi_1(t)Y_{n,m}(R/r)^{n+2}, \tag{21}$$

$$\hat{l}\tilde{\mathbf{w}} = -n(n + 1)\dot{R}\Phi_1(t)Y_{n,m}(R/r)^{n+2}, \tag{22}$$

where $Y_{n,m}$ are the spherical harmonics, $\hat{l}^2Y_{n,m} = -n(n + 1)Y_{n,m}$. The perturbation of pressure ahead of the flame front follows from Eqs. (15) and (21),

$$\tilde{P}/\rho_1\dot{R}^2 = Y_{n,m}(R/r)^{n+1}\Phi_1 \times\left[-\frac{t}{\alpha\Phi_1}\frac{d\Phi_1}{dt} - n - 2 - \frac{\alpha - 1}{\alpha} + (n + 1)\frac{\Theta - 1}{\Theta}\frac{R^3}{r^3}\right]. \tag{23}$$

As it follows from Eqs. (14) and (15) the perturbation of pressure behind the flame front satisfies the Laplace equation

$$\Delta\tilde{P} = 0\,. \tag{24}$$

Then solution of Eqs. (13)–(15) behind the flame front is

$$\tilde{P}/\rho_1\dot{R}^2 = Y_{n,m}(r/R)^n\Phi_2 \times\left[\frac{t}{\alpha\Phi_2}\frac{d\Phi_2}{dt} - n + 1 + \frac{\alpha - 1}{\alpha}\right], \tag{25}$$

$$\tilde{u}_2 = -n\Theta\dot{R}Y_{n,m}(r/R)^{n-1}\Phi_2 + \Theta Y_{n,m}\Phi_3(r)\,, \tag{26}$$

$$\hat{l}\tilde{\mathbf{w}}_2 = n(n + 1)\Theta\dot{R}Y_{n,m}(r/R)^{n-1}\Phi_2 - \Theta Y_{n,m}\left(r\frac{d\Phi_3}{dr} + 2\Phi_3\right). \tag{27}$$

Taking the perturbations of the flame front in the form

$$\tilde{R} = RY_{n,m}\Phi_4\,, \tag{28}$$

and substituting Eqs. (21)–(23), (25)–(28) into the boundary conditions, Eqs. (16)–(19), we obtain the system of ordinary equations for the unknown functions $\Phi_1, \Phi_2, \Phi_3, \Phi_4$,

$$(n+1)\Phi_1 - n\Phi_2 + \Phi_3/\dot{R} + \frac{\Theta - 1}{\Theta}\left(3\Phi_4 + \frac{t}{\alpha}\frac{d\Phi_4}{dt}\right) = 0\,, \quad (29)$$

$$\frac{t}{\alpha}\frac{d\Phi_1}{dt} + \left(1 + \frac{\alpha-1}{\alpha} + \frac{n+1}{\Theta}\right)\Phi_1 + \left(\frac{\alpha-1}{\alpha} - n + 1\right)\Phi_2 + \frac{t}{\alpha}\frac{d\Phi_2}{dt} + \frac{\Theta-1}{\Theta}\left(\frac{\alpha-1}{\alpha} + 2/\Theta\right)\Phi_4 = 0\,, \quad (30)$$

$$n(n+1)\Phi_1 + n(n+1)\Theta\Phi_2 - \frac{\Theta t}{\alpha\dot{R}}\frac{d\Phi_3}{dt} - \left(\frac{\alpha-1}{\alpha} + 2\right)\frac{\Theta\Phi_3}{\dot{R}} + n(n+1)\frac{\Theta-1}{\Theta}\Phi_4 = 0\,, \quad (31)$$

$$(n+1)\Phi_1 + \frac{t}{\alpha}\frac{d\Phi_4}{dt} + [1 + 2(\Theta-1)/\Theta]\Phi_4 = 0\,. \quad (32)$$

We look for the solution of Eqs. (29)–(32) in the form $\Phi_1, \Phi_2, \Phi_4 \sim t^{\sigma}$, $\Phi_3 \sim t^{\sigma+\alpha-1}$, where σ is the instability growth rate. Then the dispersion relation for the instability growth rate is

$$\left(\frac{\sigma}{\alpha}\right)^2(1 + n + \Theta n) + \frac{\sigma}{\alpha}[3(n+1) + d(1 + n + \Theta n) + 2n(2\Theta + n)] - \frac{\Theta-1}{\Theta}n^2(n + \Theta d) - (3\Theta + 2\Theta d + 3)n - 2n^2 + \frac{n}{\Theta} - 2 - d = 0\,, \quad (33)$$

where $d = (\alpha - 1)/\alpha$. For the case $\alpha = 1$, Eq. (34) coincides with the result obtained in [12].

The critical spherical number n_c, for which $\sigma = 0$, is determined by

$$n_c^2(n_c + \Theta d)(\Theta - 1) - 2\Theta n_c^2 + n_c - (3\Theta + 2\Theta d + 3)\Theta n_c - 2\Theta - \Theta d = 0\,. \quad (34)$$

The critical number n_c as a function of the expansion coefficient Θ is shown in Fig. 1 for a flame front with constant velocity ($\alpha = 1$) and for the experimentally observed self-accelerating flames ($\alpha = 3/2$). The stability limits are almost the same for flames with a small expansion coefficient $\Theta < 2$:$n_c \approx 2/(\Theta - 1)$. The self-similar regime $\alpha = 3/2$ becomes essentially more unstable for flames with larger expansion coefficient $\Theta > 2$. For example, for $\Theta = 20$ the stability boundary moves from $n = 10$ for the flame with constant velocity ($\alpha = 1$) to $n = 7$ for the self-similar regime with $\alpha = 3/2$.

The instability growth rate is shown in Fig. 2 versus the number of the spherical harmonic n for the flame with the expansion coefficient $\Theta = 10$. The instability growth rate for the self-similar regime of the flame propagation ($\alpha = 3/2$) differs noticeably from the increment for the flame with a constant velocity even for the case when the stability limits change slightly. For high order harmonics $n \to \infty$ the instability growth rate is proportional to the exponent of the self-similar regime,

$$\sigma \to \alpha n S/\Theta\,, \quad (35)$$

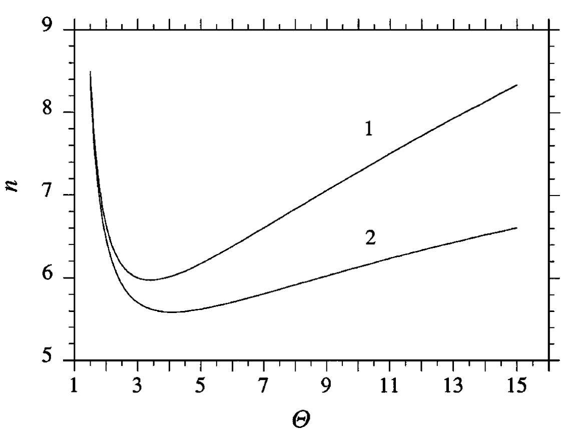

FIG. 1. Stability boundaries for a spherical flame (1) with a constant velocity; (2) propagating in the self-similar regime with $\alpha = 3/2$.

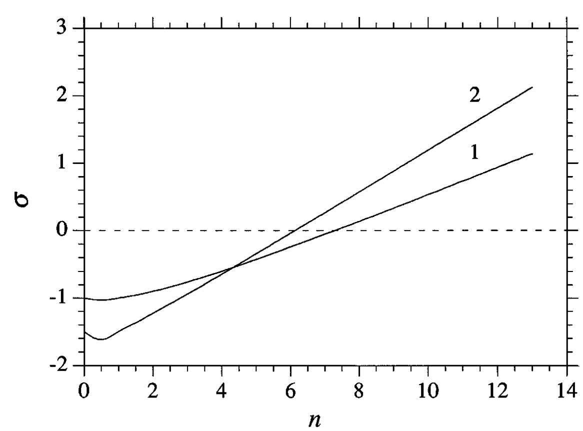

FIG. 2. The instability growth rate σ vs the spherical number n for the flame (1) with a constant velocity; (2) propagating in the self-similar regime with $\alpha = 3/2$. The expansion coefficient is $\Theta = 10$.

where

$$S = \frac{\Theta}{\Theta + 1}\left(\sqrt{\Theta + 1 - 1/\Theta} - 1\right) \tag{36}$$

is the coefficient obtained in the Landau-Darrieus theory of flame stability [1–3].

The obtained results give an explicit analytical expression for the factor A in the self-similar law, Eq. (1),

$$A = \left(\frac{2\pi\Theta u_f}{\alpha n_c \lambda_c}\right)^{\alpha} \frac{n_c \lambda_c}{2\pi}\,. \tag{37}$$

This expression follows from comparison of Eqs. (2) and (3). Thus a spherical flame with the fractal structure of the front propagates with the velocity,

$$U_f = \Theta u_f \left(\frac{2\pi R}{n_c \lambda_c}\right)^{(\alpha-1)/\alpha}, \tag{38}$$

where the critical spherical number $n_c = n_c(\Theta)$ is determined by Eq. (35). The analytical estimate for the cutoff wavelength λ_c in the case of a simple Arrhenius type reaction and neglected stoichiometry effects may be found in [7–11]. Accurate estimates of the cutoff wavelength in more general cases are available in [14].

Of course, Eqs. (37) and (38) are valid for flames unstable only against the Landau-Darrieus instability for which the self-similar regime Eq. (1) was observed (reach methane-air or hydrogen-air mixtures or lean heavy hydrocarbon-air mixtures). Additional influence of the thermal-diffusion instability makes behavior of a flame front even more complicated.

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