Roughening Transition in a One-Dimensional Growth Process

U. Alon,[1] M. R. Evans,[2] H. Hinrichsen,[1] and D. Mukamel[1]
[1]*Department of Physics and Complex Systems, Weizmann Institute, Rehovot 76100, Israel*
[2]*Department of Physics and Astronomy, University of Edinburgh, Mayfield Road, Edinburgh EH9 3JZ, United Kingdom*
(Received 6 December 1995)

A class of nonequilibrium models with short-range interactions and sequential updates is presented. The models describe one-dimensional growth processes which display a roughening transition between a smooth and a rough phase. This transition is accompanied by spontaneous symmetry breaking, which is described by an order parameter whose dynamics is nonconserving. Some aspects of models in this class are related to directed percolation in $1 + 1$ dimensions, although unlike directed percolation the models have no absorbing states. Scaling relations are derived and compared with Monte Carlo simulations.

PACS numbers: 64.60.Ak, 05.70.Fh, 05.70.Jk, 05.70.Ln

The morphology of growing interfaces has attracted much interest in recent years [1]. Many growth processes of two-dimensional surfaces exhibit a roughening transition, from a smooth phase with finite width to a rough one with diverging width. A question of interest is whether a one-dimensional (1D) growing interface with short-range interactions and unbounded noise can exhibit a roughening transition [2]. It is well known that in thermal equilibrium no such phase transition can take place as 1D interfaces are always rough. Growth processes far from equilibrium are, however, less restrictive, and the question of whether they are capable of exhibiting a roughening transition in 1D is more subtle. Most growth processes, such as those described by the Kardar-Parisi-Zhang (KPZ) equation [3], are always rough in 1D. A class of 1D models, which have a maximal velocity by which the uppermost point of the surface can propagate, has been shown to display a roughening transition [4,5]. The existence of a maximal velocity in these models is due to the use of parallel updates, and the smooth phase disappears if sequential (continuous time) updates are used. Sequential updates are a more adequate description of systems where different particles are not exactly synchronized. The question of whether a sequential update growth process is capable of exhibiting a transition from a smooth to a rough phase is still open.

A related and more general question is that of spontaneous symmetry breaking (SSB) and long-range order in 1D systems [6]. Recently a nonequilibrium 1D model with short-range interactions and unbounded noise which exhibits SSB in the thermodynamic limit was presented [7,8]. The model belongs to a class of driven diffusive systems, in which charges of two kinds are injected at both ends of a 1D lattice and are biased to move in opposite directions. The microscopic rules are symmetric under space and charge inversion. However, this symmetry is broken in the steady state of the system where the currents of the two charge species are different. In this model SSB is a result of the *conserved* dynamics of the order parameter in the bulk (charges are not created or annihilated, except at the boundaries), and the existence of *open* boundaries (two end points) at which the dynamics is different from that of the rest of the system. These two features create favorable conditions for SSB. The conserved dynamics slows down the evolution of the system; moreover, flips between the broken symmetry phase to another can originate only at the two boundary points, where the order parameter is not conserved. Simple attempts to modify the model such that either one or both of these features are eliminated results in symmetric steady states with no SSB. It would therefore be of interest to examine the possibility of SSB in 1D systems under more general conditions, namely, in homogeneous systems with periodic boundary conditions and order parameters with nonconserving dynamics.

Finally, phase transitions in homogeneous nonequilibrium 1D systems have usually been observed in the past in systems which have absorbing states (a set of states from which they system cannot escape). The canonical example is the "dry" state below the percolation threshold in directed percolation models [9–11]. Thus it is of interest to find 1D models with no absorbing states that display a phase transition.

In this Letter we present a class of nonequilibrium models with short-range interactions and sequential updates, which addresses the three questions posed above: the models exhibit a robust 1D transition which implies both the existence of a roughening transition and SSB in 1D. The dynamics is associated with a nonconserved order parameter in a homogeneous system with periodic boundary conditions. The models supply a local mechanism for eliminating islands of minority phases generated by fluctuations in the bulk of the majority phase. We derive some of the scaling properties of a particular model in this class which can be related to directed percolation [9–11]. However, unlike directed percolation, the model has no absorbing states (to be discussed below). The scaling predictions are compared to Monte Carlo simulation results.

 0031-9007/96/76(15)/2746(4)$10.00 © 1996 The American Physical Society

(a) *Model description.*—The class of models is most simply introduced in the language of interface growth [12], in which both adsorption and desorption processes take place. In the present models, desorption may take place only at the edge of a plateau. For concreteness, we study two particular models in this class, (a) a restricted solid on solid (RSOS) version that may also be considered in a charged particle representation, and (b) an unrestricted model that may be related to directed percolation. Both models are on a 1D lattice with periodic boundary conditions and are defined as follows: Let h_i be the (integer) height of the interface at site i, $i = 1, \ldots N$. The interface evolves by choosing a site i at random and carrying out one of the two following processes: (a) adsorption of an atom

$$h_i \rightarrow h_i + 1 \text{ with probability } q\,, \tag{1}$$

and (b) a desorption of atoms from the edge of a step

$$h_i \rightarrow \min(h_i, h_{i+1}) \text{ with probability } (1-q)/2\,, \tag{2}$$

$$h_i \rightarrow \min(h_i, h_{i+1}) \text{ with probability } (1-q)/2\,. \tag{3}$$

In the RSOS version, the restriction

$$|h_i - h_{i+1}| \leq 1 \tag{4}$$

is imposed at all sites. The RSOS version may be viewed as a driven diffusion model of two oppositely charged types of particles. The charges

$$c_{i,i+1} = h_{i+1} - h_i \in \{-1, 0, +1\} \tag{5}$$

are bond variables and represent a change of height between adjacent interface sites (see Fig. 1). In this representation, the dynamical rules (1)–(3) correspond to randomly selecting two neighboring bonds and performing the following processes with probabilities as indicated on the arrows:

$$\begin{array}{llll} 0+ \underset{(1-q)/2}{\overset{q}{\rightleftarrows}} +0 & & 00 \underset{1-q}{\overset{q}{\rightleftarrows}} +- & \\ -0 \underset{(1-q)/2}{\overset{q}{\rightleftarrows}} 0- & & -+ \xrightarrow{q} 00 & \end{array} \tag{6}$$

In both RSOS and unrestricted models, when q is small, smooth phase is maintained. In this phase the interface displays a small concentration of short-lived islands, and its average velocity v is zero in the thermodynamic limit. As q increases, adsorption increases and typical islands grow until, above a critical value q_c, islands merge and full new layers are completed, giving the interface a finite growth velocity. Thus, when q is small, *a local mechanism that eliminates islands is present in the model:* an island is formed with boundaries that are biased to move towards each other (due to desorption from the island edges). The evolution of a large island for $q < q_c$ is illustrated in Fig. 2. It is seen that the island shrinks, ensuring the stability of the smooth phase. This behavior is typical of islands of all sizes, except the very largest (i.e., a complete layer). The rule that no holes can be formed in a completed layer ($00 \not\rightarrow \mp$ in the RSOS version), preventing it from dividing into shrinking islands, is essential for obtaining the smooth phase. To demonstrate the existence of the roughening transition, we carried out Monte Carlo simulations of both models. In this Letter we present some of the results obtained in this study. A more detailed account will be published elsewhere [13]. The phase transition takes place at $q_c = 0.189 \pm 0.002$ for the RSOS model and $q_c = 0.233 \pm 0.001$ for the unrestricted model. The interface width is defined by the standard deviation of the height distribution $w = [N^{-1}\sum_i (h_i - N^{-1}\sum_i h_i)^2]^{1/2}$. We find that starting from a flat interface, w rises as $w \sim t^{\chi/z}$ for short times, saturating for large t at $w \sim N^{\chi}$ where N is the lattice size. At $q > q_c$, the numerical results are consistent with the KPZ exponents [3] $\chi = \frac{1}{2}$ and $z = \frac{3}{2}$, indicating a rough interface. Below q_c, w saturates at a finite value independent of N, showing that the phase is smooth. The critical behavior at $q = q_c$, shown in Fig. 3, is $w \sim \ln(N)$. In the following we discuss the symmetry breaking which takes place for $q < q_c$. We also discuss the relation to directed percolation and the critical behavior near q_c.

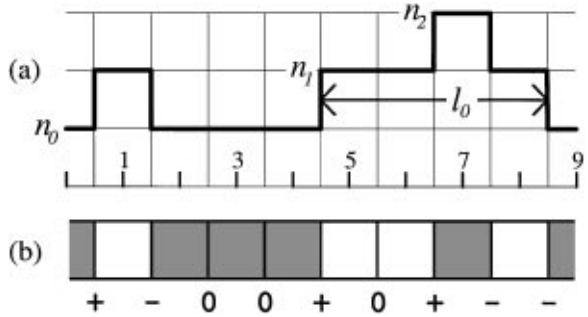

FIG. 1. (a) Typical configuration of the interface. n_k is the fraction of sites at height k above the minimal height in the configuration (here $n_0 = n_1 = \frac{4}{9}$, $n_2 = \frac{1}{9}$). The average island size grown on top of level k is l_k. (b) Mapping of the configuration of (a) to the charged particle representation, along with a site coloring, as described in text.

(b) *Spontaneous symmetry breaking.*—To demonstrate some of the model's properties, it is convenient to consider the RSOS model in the charged particle representation [Eq. (6), Fig. 1]. At $q < q_c$, the charges are arranged as closely bound $\pm$ dipoles. At $q > q_c$, the dipoles become unbound, and the fluctuations in the total charge, measured over a distance of order N, diverge with N. Thus the transition is manifested in correlations between charged particles rather than in their density. The symmetry breaking which takes place in this model is best seen by introducing a coloring scheme by which

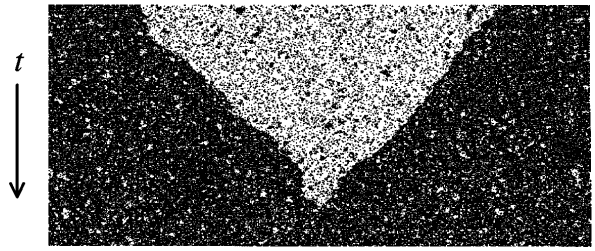

FIG. 2. Monte Carlo simulation of the RSOS model for a system of size 600 at $q = 0.130$ where $q_c = 0.189 \pm 0.002$. Each configuration is a row of pixels, with sites at even and odd heights represented by black and white pixels, respectively. Configurations at intervals of 7 moves per site (sweeps) are shown (time advances downwards) up to 2100 sweeps. At an early time, a large island of size 400 is introduced. The island shrinks and disappears, illustrating the mechanism that ensures long-ranged order at $q < q_c$. Note that small islands and islands within islands are continually generated by fluctuations, and are washed away.

each of the sites between the charged particles is colored in one of the two colors, such that the two sites adjacent to a $+$ or $-$ particle have different colors, and the two sites adjacent to a 0 particle have the same color [Fig. 1(b)]. Every move in the dynamics corresponds to a local rearrangement of the charged particles and site coloring. Under the model dynamics, any configuration of charges and coloring can evolve to any other allowed configuration of charges and coloring. Thus the model has no absorbing states. However, at $q < q_c$, a typical configuration displays unequal concentrations of the two colors. As the system evolves the configurations flip between majority colors in a time scale which was found to grow exponentially with the system size [13]. Thus, though the dynamical rules are symmetric with respect to the site colors, the system spontaneously selects one of two colors as a majority color, breaking the symmetry. The system in the phase space of charge configurations and colorings is ergodic at any finite size, but becomes nonergodic in the thermodynamic limit when $q < q_c$. To quantify this symmetry breaking, we define a magnetizationlike order parameter (valid for both the RSOS and unrestricted models)

$$M = \frac{1}{N}\sum_{i=1}^{N}(-1)^{h_i}, \tag{7}$$

which can be envisaged by considering the two colors as "up" and "down" spins. The order parameter is clearly not conserved by the dynamical rules. In the smooth phase ($q < q_c$), $\langle M\rangle \neq 0$ in the thermodynamic limit. On the other hand, in the rough phase $\langle M\rangle = 0$. Monte Carlo simulations (Fig. 3) show that near the phase transition, for both models,

$$\langle |M|\rangle \sim \epsilon^{\theta}, \quad \theta = 0.55 \pm 0.05\,, \tag{8}$$

where $\epsilon = q_c - q$.

(c) *Relation to directed percolation.*—Some features of the *unrestricted* model may be related to a directed percolation (DP) model [9–11], allowing a derivation of several scaling properties. The occupation of the lowest exposed level [n_0 in Fig. 1(a)] corresponds to the wet or percolating sites. The nonpercolating or dry sites are the sites where higher levels are occupied. A wet region may become dry at both its edges and bulk sites, while a dry region may shrink only at the edges. This defines a contact process [14], which is a sequential updated version of a DP model [10]. The percolating phase corresponds to the smooth phase in the model obtained at $q < q_c$. Thus, the occupation of the lowest level should vanish at the transition with the exponent β characterizing the critical behavior of the DP wet phase

$$n_0 \sim \epsilon^{x_0}, \quad x_0 = \beta \approx 0.28\,. \tag{9}$$

The front velocity for $q > q_c$ may be related to the lifetime of typical wet islands below the percolation threshold. These islands have a lifetime which diverges at the percolation threshold, with the critical exponent $\nu_{\|}$ of the DP coherence time [9]. The time to complete a new layer is the time it takes for its missing regions (percolation wet sites) to be covered by adsorption (dry up). Thus the velocity v is proportional to the inverse of the island lifetime:

$$v \sim (-\epsilon)^{y}, \quad y = \nu_{\|} \approx 1.73\,. \tag{10}$$

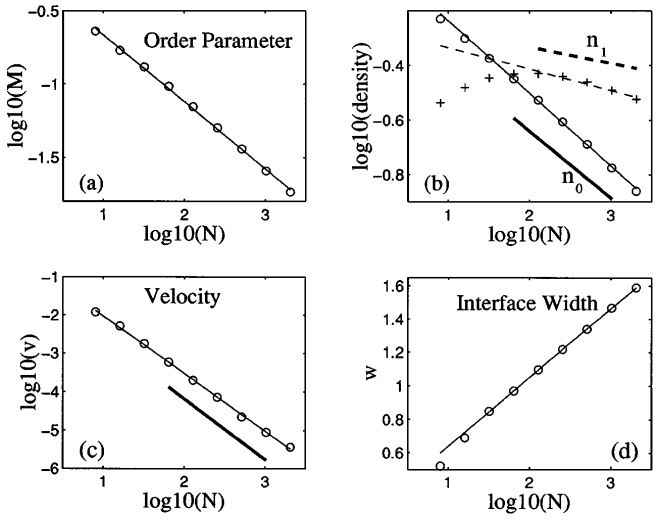

FIG. 3. Monte Carlo simulation results for the unrestricted growth model. The critical behavior of a quantity $F \sim (q - q_c)^{\alpha}$ is estimated by finite size scaling, measuring F is systems of size N at the critical point $q = q_c$, and using $F(N) \sim N^{\alpha/\nu_{\perp}}$, where $\nu_{\perp} \approx 1.10$ is the critical exponent associated with the divergence of the correlation length [see Sec. (c)]. Systems of sizes $N = 2^m$, $m = 3, \ldots, 11$ were studied at $q = q_c = 0.233$ with $2^{24} \approx 1.7 \times 10^7$ moves per site. (a) Order parameter M. (b) Density of exposed sites on the lowest exposed level, n_0, and the next level n_1. The bold lines have the slopes expected from the scaling arguments [Eqs. (9) and (11)]. (c) Interface velocity v. The bold line has the slope expected from the scaling arguments [Eq. (10)]. (d) Interface width w suggesting $w \sim \ln(N)$ (note that this graph is semilogarithmic).

These exponents are in good agreement with the values measured by Monte Carlo simulations, $x_0 = 0.29 \pm 0.03$ and $y = 1.7 \pm 0.1$, as shown in Fig. 3. For the RSOS model, there is no direct mapping to DP, since layers grown on top of a dry island affect its evolution. Monte Carlo simulations and diagonalization of the time-evolution operator of small systems ($N < 12$) [11,13] suggest that both RSOS and unrestricted models have the same exponents, and therefore belong to the same universality class.

We now present a simple scaling argument for the behavior of n_k, the density of sites at height k above the lowest exposed level. Consider first the lowest exposed level $k = 0$. According to the DP picture, there are two length scales in the problem: the average size of the dry islands, l_0, which diverges at the transition as $l_0 \sim \epsilon^{-\beta}$ [11], and the transverse correlation length $\xi_\perp \sim \epsilon^{-\nu_\perp}$ with $\nu_\perp \approx 1.10$ [11]. The two length scales are related, for a system of size N, by the finite size scaling relation $l_0 \sim \epsilon^{-\beta} f(N\epsilon^{\nu_\perp})$, where f is a function satisfying $f(s) \sim s^{\beta/\nu_\perp}$ for $s \to 0$. Similarly, $n_0 \sim \epsilon^{\beta} g(N\epsilon^{\nu_\perp})$ with $g(s) \sim s^{-\beta/\nu_\perp}$ for $s \to 0$. At criticality, one therefore has $n_0 \sim l_0^{-1} \sim N^{-\beta/\nu_\perp}$. We now consider the level $k = 1$. One may view islands of sites at heights $k \geq 2$ as growing on top of the dry islands of the $k = 1$ level, whose typical size is l_0. Applying the same scaling relations and assuming that the system size may be replaced by l_0, we find $n_1 \sim l_1^{-1} \sim l_0^{-\beta/\nu_\perp}$, where l_1 is the mean size of islands of sites with height $k \geq 2$. Repeating this reasoning for the next levels, one has $n_k \sim l_k^{-1} \sim l_{k-1}^{-\beta/\nu_\perp}$. One therefore obtains

$$n_k \sim \epsilon^{x_k}, \quad x_k = \beta(\beta/\nu_\perp)^k. \tag{11}$$

Numerical simulations (see Fig. 3) yield $x_1 = 0.07 \pm 0.02$, as compared with $x_1 \approx 0.07$ obtained from Eq. (11). More extensive numerical simulations are needed to demonstrate the validity of Eq. (11) for this exponent and the exponents associated with higher levels. A class of models that describe 1D growth processes with a roughening transition are presented. These models provide an example of spontaneous symmetry breaking that takes place with a *nonconserved* order parameter in a *ring* geometry. The zero rate of desorption of an atom whose two neighbors are on the same level is crucial for obtaining a smooth phase, and thus a roughening transition. The present models can be viewed as a hierarchy of DP-like processes, with the kth echelon DP process confined to the dry sites of the $(k-1)$-echelon DP process. The critical exponents y and x_k are found to be related to DP exponents. It would be interesting to find out whether other exponents, such as θ [Eq. (8)], may also be related to the DP problem. The models are easily generalized to higher dimensions, where the mapping to directed percolation and the scaling arguments are expected to apply. It would be of interest to construct a coarse-grained field theory [3,15] that describes the present class of models. For certain growth models mappings exist to the problem of directed polymers in a random medium [1]. It would be important to see whether such a mapping exists in the present case, and if so what is the analog of the phase transition.

We thank B. Derrida, E. Domany, G. Grinstein, C. Jayaparkash, S. Sandow, and D. Wolf for helpful discussions. This work was supported by the Minerva Foundation, Munich. M.R.E. is a Royal Society University Research Fellow.

[1] J. Krug and H. Spohn, *Solids Far From Equilibrium*, edited by C. Godrèche (Cambridge University Press, Cambridge, 1991); P. Meakin, Phys. Rep. **235**, 191 (1993); T. Halperin-Healy and Y.-C. Zhang, Phys. Rep. **254**, 254 (1995).
[2] B. Berge, L. Fauchex, K. Schwab, and A. Libchaber, Nature (London) **350**, 322 (1991).
[3] M. Karder, G. Parisi, and Y. C. Zhang, Phys. Rev. Lett. **56**, 889 (1986); E. Medina, T. Hwa, M. Kardar, and Y. C. Zhang, Phys. Rev. A **39**, 3053 (1989).
[4] R. Savit and R. Ziff, Phys. Rev. Lett. **55**, 2515 (1985).
[5] J. Kertesz and D. E. Wolf, Phys. Rev. Lett. **62**, 2571 (1989); J. Krug, J. Kertesz, and D. E. Wolf, Europhys. Lett. **12**, 113 (1990).
[6] P. Gacs, J. Comput. Syst. Sci. **32**, 15 (1986).
[7] M. R. Evans, D. P. Foster, C. Godrèche, and D. Mukamel, Phys. Rev. Lett. **74**, 208 (1995); J. Stat. Phys. **80**, 69 (1995).
[8] C. Godrèche, J. M. Luck, M. R. Evans, D. Mukamel, S. Sandow, and E. R. Speer, J. Phys. A **28**, 6039 (1995).
[9] W. Kinzel, in *Percolation Structures and Processes*, edited by G. Deutscher, R. Zallen, and J. Adler, Annals of the Israel Society Vol. 5 (Hilger, Bristol, 1983).
[10] E. Domany and W. Kinzel, Phys. Rev. Lett. **53**, 311 (1984).
[11] W. Kinzel, Z. Phys. B **58**, 229 (1985).
[12] J. M. Kim and J. M. Kosterlitz, Phys. Rev. Lett. **62**, 2289 (1989).
[13] U. Alon, M. R. Evans, H. Hinrichsen, and D. Mukamel (to be published).
[14] R. Durrett and T. M. Liggett, Ann. Probab. **18**, 186 (1981); I. Jensen and R. Dickman, J. Stat. Phys. **71**, 89 (1993).
[15] J. L. Cardy and R. L. Sugar, J. Phys. A **13**, L423 (1980).