Two-Mode Laser Power Spectra

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(Received 14 August 1995)

We investigate the properties of power spectra of a two-mode laser and predict that universal relations among the peaks of the power and noise spectra for different modes but the same frequency hold for deterministic and stochastic perturbations. These results are confirmed experimentally.

PACS numbers: 42.50.Ne, 42.55.Rz

In recent studies on multimode solid state free-running lasers (i.e., in the absence of either mode or phase locking), it has appeared that the peaks of the power spectra verify remarkable relations [1]. The purpose of this Letter is to present a study of the two-mode case for which additional and more explicit results can be obtained. We also present experimental results which confirm the theoretical analysis.

Multimode free-running solid-state lasers can be described by the Tang, Statz, and deMars rate equations [2]

$$\frac{dz_0}{dt} = w - z_0 - \sum_{n=1}^{N} \gamma_n \left(z_0 - \frac{z_n}{2}\right) I_n \,, \qquad (1)$$

$$\frac{dz_n}{dt} = \gamma_n z_0 I_n - z_n \left(1 + \sum_{k=1}^{N} \gamma_k I_k\right), \qquad (2)$$

$$\epsilon^2 \frac{dI_n}{dt} = \left[\gamma_n \left(z_0 - \frac{z_n}{2}\right) - 1\right] I_n, \qquad n = 1, 2, \ldots, N\,, \qquad (3)$$

which couple the intensity of the N modes I_n to the space average of the population inversion z_0 and the population gratings z_n. The incoherent pumping is represented by $w \geq 1$, the gain of mode n relative to the gain of the first mode is $\gamma_n \leq 1$, and ϵ^2 is the photon lifetime divided by the atomic inversion lifetime. In agreement with experimental data [3], we have assumed that ϵ is mode independent.

The essential feature which will be exploited in this Letter is that ϵ is a small parameter with typical values in the range 10^{-2} to 10^{-3}. In a previous paper [1], we have shown that the smallness of ϵ can be used to derive universal relations for the power spectrum of such lasers. What makes this derivation possible is that the linear stability analysis around the steady state is governed by complex eigenvalues with different scaling for the real and imaginary parts. The imaginary parts, which determine the oscillation frequencies Ω_j, are $O(1/\epsilon)$ while their real parts, which determine the damping rates, are $O(1)$. The main result obtained in [1] is a connection between the peak of the power spectrum $P(I_n, \Omega_j)$ for mode I_n at frequency Ω_j and the peak of the power spectrum $P(\Sigma I, \Omega_j)$ for the total intensity $\Sigma I \equiv \sum_{n=1}^{N} I_n$ at the same frequency

$$P(\Sigma I, \Omega_j) = \sum_{n=1}^{N} P(I_n, \Omega_j) + 2 \sum_{n=1}^{N} \sum_{m=1}^{n-1} \sqrt{P(I_n, \Omega_j) P(I_m, \Omega_j)} \cos(\varphi_{nmj})\,. \qquad (4)$$

This result holds in the limit $\epsilon \to 0$. It involves the parameter $\varphi_{nmj} = \theta_{nj} - \theta_{mj}$, where θ_{nj} is the phase of the nth component of the eigenvector associated with the eigenvalue λ_j whose imaginary part is Ω_j as shown in [1]. In a number of situations, the phase difference may be independent of the preparation of the system. In this case the resulting power spectrum relation becomes universal.

In the case of two modes, there are two frequencies $\Omega_L < \Omega_R$ and $\gamma_2 \equiv \gamma$. It can be shown analytically that $\varphi_{12L} = \pi + O(\epsilon)$ and $\varphi_{12R} = O(\epsilon)$. This result indicates that the low frequency Ω_L is associated with antiphase dynamics while the relaxation oscillation frequency Ω_R is associated with inphase dynamics. Therefore the power spectra equalities become

$$\begin{aligned} P(\Sigma I, \Omega_R) &= \left[\sqrt{P(I_1, \Omega_R)} + \sqrt{P(I_2, \Omega_R)}\right]^2, \\ P(\Sigma I, \Omega_L) &= \left[\sqrt{P(I_1, \Omega_L)} - \sqrt{P(I_2, \Omega_L)}\right]^2, \end{aligned} \qquad (5)$$

The universality of these relations stems from the fact that they relate peaks of different modal intensities at *the same frequency*. Hence they express a relation between different components of the same eigenvector and can therefore be independent of the preparation of the system.

We wish to determine both numerically and experimentally the extent to which the relations (5) are valid. The reason to expect a large domain of validity is that the relations (5) are derived from a linearized analysis around

TABLE I. Power spectrum normalized to $P(\Sigma I, \Omega_R)$ for the relaxation towards steady state with $\epsilon^2 = 2 \times 10^{-5}$ and $w = 2.25$. Initial condition: $w = 2.5$. (a) $\gamma = 0.99$. (b) $\gamma = 0.9$. (c) $\gamma = 0.8$.

	Ω	$\sqrt{P(I_1, \Omega)}$	$\sqrt{P(I_2, \Omega)}$	$P(\Sigma I, \Omega)$ numerical	$P(\Sigma I, \Omega)$ calculated
(a)	Ω_L	0.1283	0.1291	< 0.0001	< 0.0001
	Ω_R	0.5202	0.4798	1	1.0000
(b)	Ω_L	0.4200	0.5420	0.0149	0.0149
	Ω_R	0.7124	0.2876	1	1.0000
(c)	Ω_L	1.0105	1.6474	0.4057	0.4056
	Ω_R	0.9184	0.0816	1	1.0000

the steady state for the variables ζ_n, ζ_0, and J_n defined by $z_n = z_{ns} + \epsilon\zeta_n$, $z_0 = z_{0s} + \epsilon\zeta_0$, $I_n = I_{ns} + J_n$, and $\tau = t/\epsilon$ where the subscript s stands for steady state [4,5]. It follows from this scaling that perturbations which result in the addition of a term $O(\epsilon)$ in Eqs. (1)–(2) and a term $O(\epsilon^2)$ in Eq. (3) will not affect the relations (5). This means that correction $O(\epsilon^2)$ to z_0 and z_n and corrections $O(\epsilon)$ to I_n do not affect the relations (5).

Tests have been performed to assess the validity of the relations (5). The first test is a study of the relaxation towards a steady state [6,7]. The laser is initially in a steady state corresponding to $w = 2.5$. The pump parameter is abruptly reduced to 2.25 and the transient relaxation towards the new steady state is recorded and spectrum analyzed. The results of this numerical integration are summarized in Table I and show an excellent agreement with the relations (5): The difference between the last two columns is indeed less than ϵ.

The second test is the influence of noise. We consider a two-mode laser in the steady state and introduce an external source of noise. We added to the right hand side of the modal intensity equation (3) a source term of the form $\alpha I_{ns}\zeta(t)$ where $\alpha = 0.05$ is the noise amplitude, I_{ns} is the steady intensity of the nth mode ($n = 1$ or 2), and $\zeta(t)$ is a sequence of random numbers uniformly distributed on the interval $[-1, 1]$. The time evolution is spectrum analyzed and the result is displayed in Table II. Here again, the agreement between the numerical value of $P(\Sigma I, \Omega)$ and the result obtained using the relations (5) is excellent, with a discrepancy which is always lower than ϵ.

An experimental verification of the power spectrum relation (5) has been carried out by using a microchip $LiNdP_4O_{12}$ (LNP) laser oscillating in the two-mode regime. The LNP laser consists of a 1 mm thick crystal with dielectric mirrors coated on both surfaces and it is pumped by an Ar laser. The oscillation wavelength was 1.32 μm and the TEM_{00} mode lasing threshold was 250–260 mW depending on the pump position on the crystal. By changing the pump position, the pump power for the onset of the second cw lasing mode (and therefore γ) was changed resulting from a slight change in the cavity length (e.g., crystal thickness). We tested the relations (5) at two different pump positions. The second threshold w_c was 1.185 and 1.260, respectively.

An example of power spectra averaged over a long period of time for modal and total intensities of the free-running LNP laser driven by "white" noise is shown in Fig. 1, where the pump power is $w = 3.27$ and $w_c = 1.26$. Results obtained from different pump and gain are summarized in Table III, together with calculated values based on the measured power spectral intensities using (5). Experimental results are found to verify the universal relation (5) of two-mode laser power spectra excellently.

Thus we have shown analytically, numerically, and experimentally that the relations (5) have a large domain of applicability. Additional tests have been made with gain and loss modulation. They also verify the relations (5) provided that the modulation amplitude is less than or equal to ϵ^2.

The simplicity of the two-mode case is related to the existence of only two internal frequencies, Ω_L and Ω_R,

TABLE II. Noise spectrum normalized to $P(\Sigma I, \Omega_R)$ with $\epsilon^2 = 2 \times 10^{-5}$, $w = 2.25$ and noise level 5×10^{-2}. (a) $\gamma = 0.99$. (b) $\gamma = 0.9$. (c) $\gamma = 0.8$.

	Ω	$\sqrt{P(I_1, \Omega)}$	$\sqrt{P(I_2, \Omega)}$	$P(\Sigma I, \Omega)$ numerical	$P(\Sigma I, \Omega)$ calculated
(a)	Ω_L	1.4238	1.4714	0.00295	0.00226
	Ω_R	0.5285	0.4728	1	1.0026
(b)	Ω_L	1.4633	1.8197	0.1287	0.1270
	Ω_R	0.7212	0.2804	1	1.0032
(c)	Ω_L	0.4760	0.7538	0.0781	0.0772
	Ω_R	0.9248	0.0756	1	1.0008

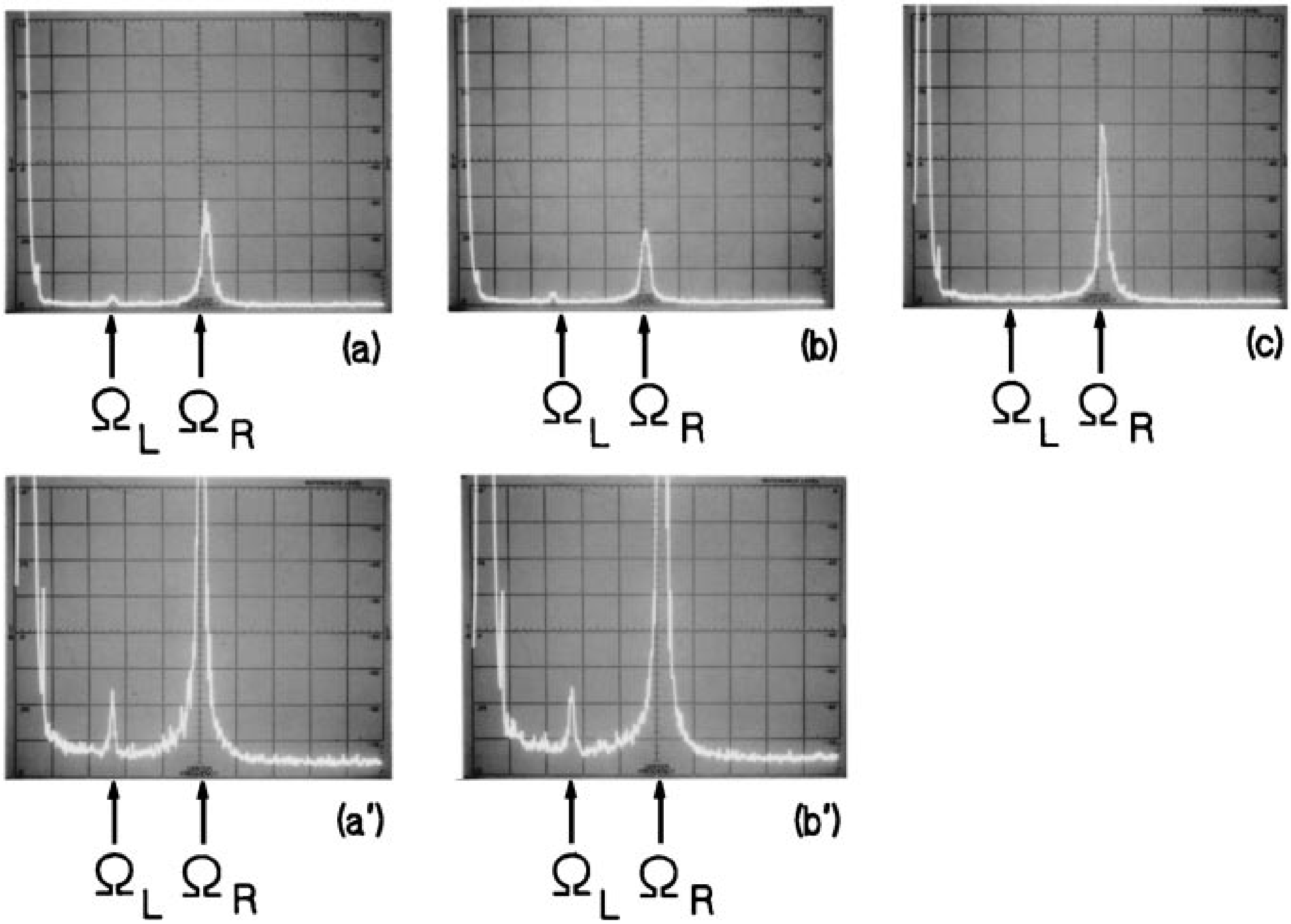

FIG. 1. Two-mode LNP laser power spectra in the free-running condition. (a) Power spectrum for mode I_1. (a′) Enlarged I_1 spectrum. (b) Power spectrum for mode I_2. (b′) Enlarged I_2 spectrum. (c) Power spectrum for the total intensity $I_1 + I_2$, where the vertical scale is twice that of (a) and (b). The horizontal scale is 100 kHz/div.

which are determined by solving a quadratic equation. In the case of three or more nondegenerate modes, there is practically no analytic information which can be obtained. Using the results of [4], one finds that for $N = 2$ additional relations between the power spectra can be derived analytically. In particular, we have

$$P(I_2, \Omega_R)/P(I_1, \Omega_R) = S_+^2,$$
$$P(I_2, \Omega_L)/P(I_1, \Omega_L) = S_-^2, \tag{6}$$

$$S_\pm = \frac{(\gamma - 1)(4 - \gamma z^2) \pm \sqrt{[(\gamma - 1)(4 - \gamma z^2)]^2 + 16[1 - \gamma(z - 1)]^2(z - 1)(\gamma z - 1)}}{4\gamma(z - 1)[1 - \gamma(z - 1)]} \tag{7}$$

TABLE III. Observed noise spectrum normalized to $P(\Sigma I, \Omega_R)$ of free-running two-mode LNP laser. (a) $w = 1.6$, $w_c = 1.18$. (b) $w = 1.55$, $w_c = 1.26$. (c) $w = 3.27$, $w_c 1.26$.

	Ω	$\sqrt{P(I_1, \Omega)}$	$\sqrt{P(I_2, \Omega)}$	$P(\Sigma I, \Omega)$ numerical	$P(\Sigma I, \Omega)$ calculated
(a)	Ω_L	0.22	0.26	< 0.01	< 0.01
	Ω_R	0.56	0.44	1	1.00
(b)	Ω_L	0.26	0.27	< 0.01	< 0.01
	Ω_R	0.54	0.43	1	0.94
(c)	Ω_L	0.16	0.16	< 0.01	< 0.01
	Ω_R	0.55	0.46	1	1.02

with $\gamma \equiv \gamma_2$ and z is the steady state value of z_0. Therefore, the ratios $P(I_2, \Omega_R)/P(I_1, \Omega_R)$ and $P(I_1, \Omega_L)/P(I_2, \Omega_L)$ increase with w at constant γ. For a constant pump w, there is a critical value of the gain γ_c below which the laser is single mode. The three peaks $P(I_1, \Omega_L)$, $P(I_2, \Omega_R)$, and $P(I_2, \Omega_L)$ vanish for $\gamma = \gamma_c$. However, the analytical expression (7) implies $9/4 < P(I_2, \Omega_L)/P(I_1, \Omega_L) < 4$ in the limit $0 < \gamma - \gamma_c \rightarrow 0$. For instance, $\gamma_c \simeq 0.76$ for $w = 2.25$, and Table I gives $P(I_2, \Omega_L)/P(I_1, \Omega_L) = 2.6578$ for $\gamma = 0.8$.

An essential property of the relations (5) is that they are independent of the parameters of the system. In contrast, the relations (6) require an explicit knowledge of all the parameters. We have verified that the relations (6) are in agreement with the numerical results presented in Tables I and II.

This research was supported in part by the Fonds National de la Recherche Scientifique (Belgium) and the Inter-University Attraction Pole program of the Belgian government. K. O. is indebted to T. Yamada of NTT Basic Research Laboratories for supplying an LNP crystal.

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