Classical Chaos as an Environment for Dissipation

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We study an isolated three-dimensional system composed of a slow one-freedom system of interest interacting with a soft-chaotic environment. We present numerical evidence that the averaged motion of the slow system is dissipative, though not exactly governed by the Langevin equation. An analytical reasoning is proposed to explain the results, circumventing the unsufficiency of the adiabatic hypothesis.

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This Letter examines the question of modeling dissipation by coupling of a macroscopic system to a fast, low-dimensional, and chaotic environment. This problem was first investigated by Wilkinson [1], who found that adiabatic time-dependent chaotic Hamiltonians can be a source of dissipation. A general adiabatic theory has been provided by Berry and Robbins (BR) [2], who considered explicitly a microcanonical ensemble of chaotic systems in order to define the averaged dynamics of the macroscopic system. It was shown that, for a sufficiently slow macroscopic motion, the adiabatic invariance of the energy surface [3,4] gives rise to a Born-Oppenheimer reaction force, and the first correction to it is a force proportional to the velocity. This force in turn is split into a so-called "geometric magnetic" and a "deterministic" friction. The time dependence of the Hamiltonian presupposes the slow motion keeps its external dynamics (frozen) while coupled, as stated in BR's work. This is somewhat problematic when we have to define the slow motion's energy, whose decrease is stated at last. Further, once the energy of the slow motion has flown to the chaotic system, how long does it stay in it? These questions cannot be addressed when we start with an explicit time dependence on the chaotic Hamiltonian.

An alternative way of modeling dissipation was developed in the early 1980s by Caldeira and Leggett (CL) [5]. In their model, a macroscopic "system of interest" is coupled to a reservoir of harmonic oscillators. Using a coordinate-coordinate coupling between the macroscopic system, here a one-dimensional system whose coordinate is named z, and the reservoir, it was found that the former motion is governed by the classical Langevin equation

$$M\ddot{z} + \eta\dot{z} + \frac{\partial V}{\partial z} = f(t)$$

provided the set of oscillators has a linear spectral distribution of frequencies. As pointed out in [1], this approach is based on a thermodynamical limit, for the number of oscillators tends to infinity.

Our procedure removes the inconvenience of time-dependent Hamiltonians by constructing a weakly coupled universe $S \tilde{+} R$, an idea borrowed from the CL formalism, whose parts are a system of interest, a slow system S, and its environment, a smooth chaotic system R. By using a representation of the chaotic system in terms of the eigenfunctions of the corresponding Liouville operator [6], we establish a connection between the BR and the CL formalisms.

We consider an isolated system governed by the Hamiltonian

$$H = H_N + H_I + H_z, \tag{1}$$

where

$$H_N(x, y, p_x, p_y) = \frac{p_x^2}{2} + \frac{p_y^2}{2} + \left(y - \frac{x^2}{2}\right)^2 + \frac{\mu}{2}x^2,$$

$$H_I(x, z) = \gamma x z,$$

$$H_z(z, p_z) = \frac{p_z^2}{2M_z} + \frac{\lambda}{2}z^2 + \beta z^4.$$

H_N is the Hamiltonian for the known Nelson potential, a system which is fairly regular for low energies, $E \lesssim 0.05$ but chaotic for higher ones, $E \gtrsim 0.3$ and mixed for intermediate values [7]. The parameter μ is set equal to 0.1 throughout The property of soft chaoticity is particularly useful in the present application, since we want to compare the behavior of the slow system's energy for the two regimes. The dynamical variable z varies slowly for $\beta/M_z \ll 1$, if $\lambda = 0$, or for $\lambda/M_z \ll 1$, if $\beta \gtrsim 0$, and therefore can play the role of an adiabatic parameter for the Hamiltonian function $h(z) = H_N + H_I$.

We first calculate numerically the reaction force produced by the fast chaotic system on the slow motion. Given an initial point on the phase space of the slow system, we wish to determine in particular whether its evolution under the total Hamiltonian H will increase or decrease its energy. The fluctuations due to a specific choice of initial condition on the chaotic system are rubbed out by averaging over the *restricted* microcanonical distribution

$$\rho(z, p_z) = \frac{\delta(E - H)}{\Sigma(E, z)}, \tag{2}$$

where the normalization $\Sigma(E,z)$ is the derivative with respect to energy of $\Omega(E,z) = \int \Theta(E - H)\, dx\, dy\, dp_x\, dp_y$.

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To generate an initial ensemble numerically, we parametrize the energy surface of the fast system with three angles [8]. The volume inside the energy shell of the chaotic system, Ω, is the adiabatic invariant [3]. We can easily evaluate Ω for the Nelson potential, because its Hamiltonian is a sum of squares, obtaining $\Omega = \pi^2\sqrt{\frac{2}{\mu}}[E - H_z + \frac{\gamma^2 z^2}{2\mu}]^2$. From the relation $\Omega =$ const we derive the adiabatic approximation for the slow system's energy share during the evolution of the composed system:

$$H_z - \gamma^2 z^2/2\mu \approx \text{const},$$

which means that $-\frac{\gamma^2 z^2}{2\mu}$ is the renormalizing Born-Oppenheimer (BO) potential V_r due to coupling. It is important to note that the BO term does not depend on the state of the fast system, regular or chaotic might it be.

The actual deviation from an adiabatic trajectory leads to higher order corrections. Brown, Ott, and Grebogi have studied numerically the diffusion behavior of the adiabatic invariant $\Omega(z(t), p_z(t))$ [4], another manifestation of the dissipation we are expected to measure for the energy of the slow system.

We calculate the mean values of the position and momentum of the slow system for 1.8 period of its own (decoupled) motion. Explicitly, we plot $(\langle z(t)\rangle, \langle p_z(t)\rangle)$, where $\langle\cdot\rangle = \int \rho \cdot dx\, dy\, dp_x\, dp_y$ and $z(t), p_z(t)$ are exact solutions of the full Hamiltonian (1) for fixed z_0 and p_{z_0} and each initial condition of the chaotic system. We attribute $E_z \equiv H_z(\langle z\rangle, \langle p_z\rangle)$ to the energy of the slow system and $E_{zr} = H_z(\langle z\rangle, \langle p_z\rangle) - V_r(\langle z\rangle)$ to its renormalized energy.

We set the parameters $\gamma = 6e-3$, $\lambda = 0$, $\beta = 4e-4$, and $M_z = 100$. The period of the decoupled one-dimensional system is of the order of 1000 for these settings, much bigger than the shortest periodic orbit of the Nelson potential, namely, $\sqrt{2}\,\pi$. The method of numerical integration is an implementation of the sixth order simplectic integrator [9], which guarantees that the total energy is conserved, a property not shared by the family of Runge-Kutta integrators. Its variation is less than 10^{-7} for the whole evolution.

For $E_{z_r}(z_0, p_{z_0}) = 0.002\,87$ and the total energy $E = 0.04$, so that the Nelson potential is in a regular region, the slow motion is still periodic. We conclude from Fig. 1 that its energy has been renormalized due to coupling, and E_{z_r} remains fairly constant, as expected.

For the total energy $E = 0.38$, two phase space averaged trajectories of the slow motion are depicted in the inset of Fig. 2, showing clearly that its initial energy goes to the chaotic system. How this energy is carried out by the chaotic system is shown in Fig. 2. We verify that the oscillations of E_z are bigger than the renormalizing potential, also depending on the initial condition z_0, p_{z_0} being at an even or odd quadrant.

These results cannot be understood in light of the adiabatic approximation alone, because if it were then the energy E_{z_r} would decrease monotonically. To explain the departure from a purely or mostly dissipative behavior of the renormalized energy E_{z_r}, we shall consider the problem from a different point of view, thereby establishing a bridge to the bath of oscillators formalism [5].

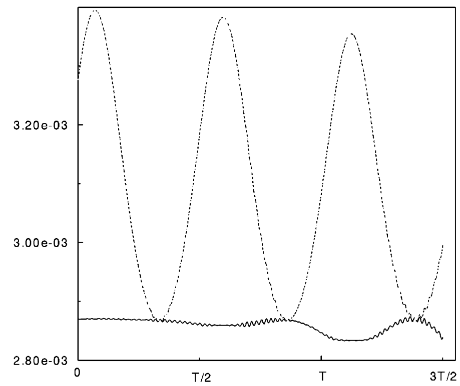

FIG. 1. The energy of the slow motion, E_z, as a function of time is the oscillating curve, while the renormalized energy $E_{z_r} = E_z - V_r$ is fairly constant on this scale. Total energy 0.04 sets the Nelson potential in a regular regime.

We write the set of Hamilton equations for the chaotic system, for the moment not considering the coupling, in terms of the Liouville operator for a complex valued dynamical variable f

$$\dot{f}(q,p) = [f, H] =: H : f = Lf, \tag{3}$$

and envisage it in connection to the eigenvalue equation for the operator L.

We shall henceforward use the Hilbert space notion in classical mechanics. We start by considering the set of all C^∞ dynamical variables, and the distributions obtained by the restriction of those to the maximal invariant manifolds: periodic orbits or families of them (tori). A measure $d\mu$ concentrated on the periodic orbits (or tori), with Dirac deltas, can be defined. Notice that the wandering set is ascribed zero measure.

The Hilbert space we consider is $L^2(\mathbf{R}^{2N}, d\mu)$, a notation which also defines the inner product, $\langle f, g\rangle = \int f\,\overline{g}\,d\mu$. As discussed in Santilli [6], if the Hamiltonian

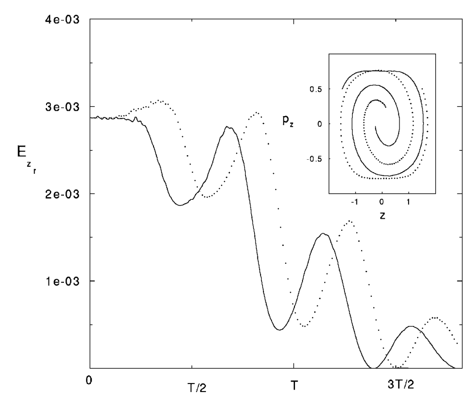

FIG. 2. Renormalized energy $E_{z_r}(\langle z\rangle, \langle p_z\rangle)$ as a function of time. The amplitude of the oscillations is larger than the BO potential. Inset: phase space plots of the averaged slow motion, initial conditions $(z, p_z) = (\pm 1.5, 0.5)$. Total energy 0.38 sets the Nelson potential in a chaotic regime.

possesses global solutions for any initial condition, the operator $-iL$ is essentially self-adjoint, which means there is a unique extension of $-iL$ that is self-adjoint [11]. Global solutions are guaranteed for the Nelson potential because it is bounded from below. The Hilbert space of dynamical variables in which $-iL$ is self-adjoint contains $L^2(\mathbf{R}^{2N}, d\mu)$. In it, we have the associated Hamilton's equations formally linearized, but by now we have to deal with an infinite set of them, $-iL\varphi_k = \lambda_k \varphi_k$,

$$d\varphi_k/dt = L\varphi_k = i\lambda_k\varphi_k\,, \tag{4}$$

where λ_k are real. We can separate the real and imaginary parts of φ_k, $\varphi_k = u_k + iv_k$, obtaining $\frac{du_k}{dt} = -\lambda_k v_k$ and $\frac{dv_k}{dt} = \lambda_k u_k$ and $\frac{dv_k}{dt} = \lambda_k u_k$. To find H_N in this basis, we must fulfill the relations $[u_l, H_N] = -\lambda_l v_l$ and $[v_l, H_N] = -\lambda_l u_l$. Therefore we can write H_N as $\sum_k \frac{\lambda_k}{2}(u_k^2 + v_k^2)$.

From the form of Eq. (4), it is clear that φ_k are periodic with period $1/\lambda_k$. It follows that the eigenfunctions of the Liouville operator are closely related to the periodic orbits, in the ergodic case, or with the rational tori, in the integrable case. Each of them resides in a specific invariant manifold, vanishing outside. Their proportionality to the characteristic function of an invariant manifold carries the information about the local variables perpendicular to this manifold, action variables in the integrable case. The set of eigenfunctions are, in fact, the union of Fourier bases over each periodic orbit. Let $\{\gamma_k, k = 1, 2, \ldots\}$ be an ordering of the periodic orbits. Then, in the chaotic case,

$$\varphi_{kn}(q,p) = \frac{1}{\sqrt{a(\gamma_k)}}\, e^{i\lambda_{kn}t}\chi_{\gamma_k}\,, \quad \lambda_{kn} = \frac{2\pi n}{T_k}\,, \tag{5}$$

where T_k is the period of the orbit γ_k, t is a parameter along this orbit, χ_{γ_k} is its characteristic function, and $a(\gamma_k)$ is a positive weight of the orbit. The form of the eigenfunctions, so simple and quite independent of the specific potential, hides the complexity of chaotic dynamics, contained in the weights $a(\gamma_k)$ and the Dirac delta measure. All we need here is that these eigenfunctions form an orthonormal basis, and the details of this Hilbert space approach will be published elsewhere [10].

Recall that $f(x, y, p_x, p_y) = x$ is a dynamical variable. Since the set $\{\varphi_k\colon k = 1, 2, \ldots\}$ is an orthonormal basis for the dynamical variables, we may write

$$x = \sum_k C_k \varphi_k = \sum_k A_k u_k + B_k v_k\,, \tag{6}$$

where A_k and B_k are real. The coefficients C_k are determined by the usual relation $C_k = \langle x, \varphi_k\rangle$.

We write a formal Hamilton function associated with the coupled set of equations

$$\mathcal{H} = \sum_k \frac{\lambda_k}{2}(u_k^2 + v_k^2) + \gamma\sum_k (A_k u_k + B_k v_k)z + H_z\,,$$

which gives an infinite system, linear for Nelson's potential part. Applying the Laplace transform on the perturbed equations for $u_k(t)$ and $v_k(t)$ and substituting the results in the Newtonian equation for the slow motion we get

$$\begin{aligned} M_z\ddot{z} + \lambda z + 4\beta z^3 = -\gamma\Big\{&\sum A_k[u_k(0)\cos\lambda_k t - v_k(0)\sin\lambda_k t] + \sum B_k[v_k(0)\cos\lambda_k t + u_k(0)\sin\lambda_k t]\Big\} \\ &+ \gamma^2\sum(A_k^2 + B_k^2)\lambda_k\mathcal{L}^{-1}\left(\frac{\tilde{z}(s)}{s^2 + \lambda_k^2}\right), \end{aligned} \tag{7}$$

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform and $\tilde{z}(s)$ is the Laplace transform of $z(t)$.

The first two sums on the right hand side can be grouped and simplified as $-\gamma\sum[A_k u_k(t) + B_k v_k(t)] = -\gamma x_d(t)$, where $x_d(t)$ denotes a particular solution of the decoupled motion. This is a fluctuating force whose average over the restricted microcanonical distribution generates the BO force. To see this, expand the microcanonical distribution in a power series of the coupling constant γ:

$$\rho \approx \frac{\delta(E - H_N - H_z)}{\Sigma(\gamma)} - \gamma xz\,\frac{\partial_E\delta(E - H_N - H_z)}{\Sigma(0)}\,,$$

so that $\langle x_d(t)\rangle = \int \rho x\, dx\, dy\, dp_x\, dp_y = \frac{\gamma\langle z\rangle}{\mu} + o(\gamma^3)$, where we write $\Sigma = \Sigma(\gamma)$ to emphasize its dependence on the coupling constant. The next order nonvanishing term of $-\gamma\langle x_d(t)\rangle$ is proportional to γ^4 and we will neglect it. This is, in fact, a weak coupling limit, also present in the CL formalism when one considers the perturbation each oscillator of the bath suffers.

The average of the last term can be written in terms of the autocorrelation function $C_x(\tau) = \langle x_d(t+\tau)x_d(t)\rangle$. Defining the spectral density [12]

$$S_x(\omega) = 4\,\mathrm{Re}\int_0^\infty e^{-i\omega\tau}C_x(\tau)\,d\tau\,, \tag{8}$$

and using the expansions $x(t+\tau) = \sum_l e^{+i\lambda_l(t+\tau)}C_l^* \times \varphi_l^*(0)$ and $x(t) = \sum_k e^{-i\lambda_k t}C_k\varphi_k(0)$, we get

$$\begin{aligned} S_x(\omega) &= 4\,\mathrm{Re}\sum_{k,l}\int_0^\infty e^{-i(\omega-\lambda_l)\tau - i(\lambda_k-\lambda_l)t}\langle C_k C_l^*\varphi_k\varphi_l^*\rangle\,d\tau \\ &= 4\pi\sum_l \delta(\omega - \lambda_l)|C_l|^2 \end{aligned}$$

by the orthonormalization of the basis. Therefore we can write the last term in Eq. (7) as

$$\frac{\gamma^2}{4\pi}\int_0^\infty S_x(\omega)\left(\int_0^t \sin\omega(t-t')z(t')\,dt'\right)d\omega\,. \tag{9}$$

The form of this term is in exact agreement with the Caldeira-Leggett treatment of dissipation. But here we have a spectral density which depends on the chaotic motion. In particular, we can show [see Eq. (5) and (9)] that the long periodic orbits in a chaotic system cause

$S_x(\omega)$ to be nonvanishing for low frequencies, a property that is in contrast with the linear dependence the spectral density is assumed to have in [5].

We display in Fig. 3 the numerically calculated (decoupled) correlation $C_x(\tau)$ for the total energy $E = 0.38$. It can be well fitted by the usual correlation $\sigma^2 e^{-\alpha t} \cos\omega_0 t$ [12], where $\sigma^2 = 1.886$, $\alpha = 4.17 \times 10^{-2}$, and $\omega_0 = 0.1966$. Using this expression, $S_x(\omega)$ is given by

$$2\alpha\sigma^2\left(\frac{1}{(\omega + \omega_0)^2 + \alpha^2} + \frac{1}{(\omega - \omega_0)^2 + \alpha^2}\right).$$

Unfortunately, the integral in ω cannot be written in closed form, as it has an odd integrand, but the evolution equation we have for $\langle z\rangle$ can still be handled numerically. Considering the equation for $\langle z\rangle$ we obtained

$$M_z\langle\ddot{z}\rangle + \lambda\langle z\rangle + 4\beta\langle z^3\rangle + \left\langle\frac{\partial V_r}{\partial z}\right\rangle = \mathcal{F}(z), \qquad (10)$$

where $\mathcal{F}$ stands for the convolution term depending on the correlation function (9), we see that for nonzero β the term in $\langle z^3\rangle$ turns out to be a time-dependent potential for the averaged motion. Its difference from $\langle z\rangle^3$ is large when chaos is present, while tiny when the fast variables are in a regular regime. It also increases with time and is the source of enormous oscillations of the averaged renormalized energy E_{z_r}, shown in Fig. 2.

A simplified application of this approach is to consider $\lambda = 2 \times 10^{-3}$ and $\beta = 0$, a setting that makes the period of the (decoupled) slow motion $\approx$1400 and linearizes (10). We integrate this equation iteratively, comparing the curve for the renormalized energy E_{z_r} obtained from it with the one obtained from the previous method: coupling to a microcanonical ensemble of chaotic systems. The curves are shown in Fig. 4. Besides the good agreement, attention must be drawn to the amplitude of variation of E_{z_r}, much lower than both the observed in Fig. 2 and the BO potential $\gamma^2\langle z\rangle^2/2\mu$.

Therefore, when the system of interest is linear, the Born-Oppenheimer force is dominant and dissipation appears as a small correction. However, if the z system has a nonlinear character, exemplified here by a quartic oscillator, the difference $\langle\frac{\partial V}{\partial z}\rangle - \frac{\partial}{\partial z} V(\langle z\rangle)$ prevails over the BO force for the chaotic regime. Dissipation, nevertheless, still exists in the latter case.

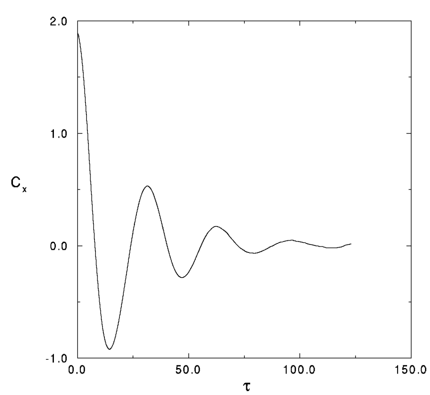

FIG. 3. Autocorrelation of the fast coordinate x as a function of time. The correlation time $\frac{2}{\pi\alpha} \approx 15$ is a measure of the fastness of the chaotic motion.

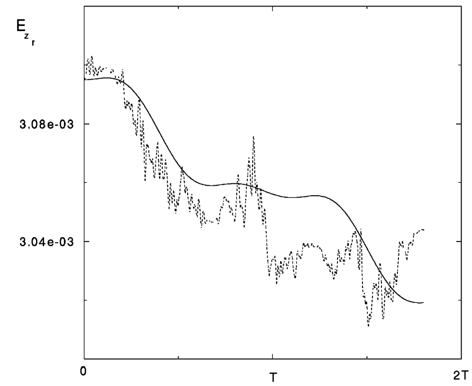

FIG. 4. Renormalized energy E_{z_r} in function of time for a slow linear oscillator, $\lambda = 2e - 3$, $\beta = 0$. Smooth curve obtained from Eq. (9) and the other obtained from the averaged motion. Total energy is the same as in Fig. 2. The amplitude of variation of E_{z_r} is less than the BO potential.

The conclusion of this work can be summarized in Eq. (10), which shows a second effect of the coupling of a system of interest to a chaotic environment besides the Born-Oppenheimer force, a term depending only on the correlation of the coupled chaotic coordinate.

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