Relationship between Delayed and Spatially Extended Dynamical Systems

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The interpretation of delayed dynamical systems (DDS) in terms of a suitable spatiotemporal dynamics is put on a rigorous ground by deriving amplitude equations in the vicinity of a Hopf bifurcation. We show that comoving Lyapunov exponents can be defined and computed in a DDS. From the propagation of localized infinitesimal disturbances in DDS, we show the existence of convective type instabilities. Moreover, a widely studied class of DDS is mapped onto an evolution rule for a spatial system with drift and diffusion.

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The investigation of dynamical systems with delayed feedback such as

$$\dot{y} = \mathcal{F}(y, y_d), \tag{1}$$

where $y_d \equiv y(t - T)$ is the delayed variable and T is the time delay, has revealed analogies with 1D spatially extended systems (SES). It has been shown that statistical indicators such as, e.g., the fractal dimension, are extensive quantities, proportional to the delay time T, which appears to play a role very similar to the size of a spatial system [1]. A more direct evidence of the analogy has been found by introducing a two-variable representation, i.e., by defining the time as

$$t = \sigma + \theta T, \tag{2}$$

where $\sigma \in [0, T]$ is interpreted as a space variable and $\theta \in \mathcal{N}$ plays the role of a (discrete) time [2]. In fact, such a representation allows identifying the formation and propagation of "space-time" structures as, e.g., defects and spatiotemporal intermittency [3].

The advantage of a space-time representation is evident once it is realized that the long-range interaction with the delayed variable can be reinterpreted as a short-range coupling in the new variables, since $y_d = y(\sigma, \theta - 1)$. However, at variance with a SES, the variable y is here updated asynchronously. Moreover, there is a clear difference in the boundary conditions which connect each delay unit with the following one. In the limit $T \rightarrow \infty$, we expect the latter difference to play no significant role in determining the "bulk" properties of a delayed dynamical system (DDS). This is confirmed by numerical simulations revealing that the Lyapunov spectrum is independent of T in the limit of large delays [4].

A general question arising from the above observations is to what extent the behavior of a delayed system can be assimilated to that of a SES and its properties therefrom explained. In this Letter, we show the existence of a deep relationship between DDS and asymmetric SES. In particular, instabilities arising in DDS can be "generally" interpreted as convective instabilities. This analogy is first proved by extending the method of comoving Lyapunov exponents [5] to DDS. From numerical and analytical studies, it turns out that only disturbances propagating within a suitable angular cone are exponentially amplified.

Moreover, we apply the method of amplitude equations in the vicinity of a Hopf bifurcation, arriving at a complex Ginzburg-Landau equation. Such an approach applies, for instance, to the experimental results of Ref. [3] which have been successfully compared with the evolution equation for the complex variable y,

$$\dot{y} = \mu y - (1 + i\beta)|y|^2 y + \eta y_d. \tag{3}$$

Finally, we consider the well known class of systems

$$\dot{y} = -y + F(y_d), \tag{4}$$

which, for the particular choices $F(z) = bz/(1 + z^{10})$ and $F(z) = a \sin(z - z_0)$, reduces to Mackey-Glass [6] and Ikeda [7] models, respectively. We show that the interpretation of Eq. (4) as the composition of a local discrete-time nonlinear mapping with a diffusion operator is very powerful indeed.

Let us start from the propagation of localized disturbances in a generic DDS such as in Eq. (1). The linear stability analysis amounts to studying

$$\dot{u} = \mu u + \eta u_d, \tag{5}$$

where $u = \delta y$, $\mu = \partial_y \mathcal{F}$, and $\eta = \partial_{y_d} \mathcal{F}$ can be assumed to be complex. In the (σ, θ) plane, Eq. (5) can be rewritten as

$$\partial_\sigma u(\sigma, \theta) = \mu u(\sigma, \theta) + \eta u(\sigma, \theta - 1). \tag{6}$$

Computing comoving Lyapunov exponents is tantamount to finding a solution of Eq. (6) with initial condition $u(\sigma, 0) = \delta(\sigma)$, where $\delta(\sigma)$ is the Dirac δ function. The comoving Lyapunov exponent is then defined as

$$\Lambda(\alpha) = \lim_{r \rightarrow \infty} \frac{\ln|u(\sigma, \theta)|}{r}, \tag{7}$$

where r and α are the polar coordinates in the (σ, θ) plane. The above definition is slightly different from the usual one [5] in that the spectrum Λ is parametrized by α instead of by $v = \tan\alpha$, while the growth rate Λ is referred to the 2D distance r instead of the time θ. This choice is motivated

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by the desire to describe the propagation with infinite velocity ($\alpha = \pi/2$) also in a properly scaled manner.

Some general features can be extracted from the study of the simple case μ and η constant (i.e., by investigating the stability of the stationary state). The exact solution of Eq. (6) is

$$u(\sigma, \theta) = \frac{\eta^{\theta}}{(\theta - 1)!} \sigma^{\theta - 1} e^{\mu \sigma}. \tag{8}$$

In the limit of large σ and θ, we can make use of Stirling's approximation, obtaining

$$\Lambda(\alpha) = \mu_R \sin\alpha + [1 + \ln(|\eta| \tan\alpha)] \cos\alpha , \tag{9}$$

where μ_R indicates the real part of μ, while $|\cdot|$ indicates the modulus. Analogously to SES, where the comoving exponent is independent of the system size, here $\Lambda(\alpha)$ is independent of the delay T. The maximum growth rate is attained at an angle $\alpha = \alpha_0$ with the solution of the transcendental equation

$$v_0^2 \ln(|\eta| v_0) = \mu_R v_0 + 1 , \tag{10}$$

where $v_0 = \tan(\alpha_0)$ is the corresponding velocity of propagation. The threshold condition for an exponential instability, $\Lambda_{\max} = 0$, yields $|\eta| = -\mu_R$ which is derived for $v_0 = 1/|\eta|$ [8]. It is interesting to notice that the stability is determined from the *real part* of μ as it normally happens in continuous-time flows, and from the *modulus* of η which plays the role of a multiplier in a discrete-time mapping.

In the limit $\alpha \to 0$, the leading contribution to Λ comes from the logarithmic term which becomes infinitely negative, no matter how large the values of both μ and η are. This means that there cannot be any propagation of disturbances with zero velocity as indirectly confirmed by the experimental results of Ref. [3] and by the simulations reported therein, where coherent structures are always associated with a nonzero finite drift. At the opposite limit we find that the growth rate corresponding to a infinite velocity is μ_R, i.e., it depends on the "instantaneous coupling" only.

In order to test the general validity of the above findings, we have computed numerically several spectra of comoving exponents for models (3) and (4). Whether or not the maximum of the spectrum is strictly positive, one can notice in Fig. 1 that it always occurs at a finite angle, meaning that there is a preferred direction (i.e., velocity) for the propagation of perturbations.

SES exhibit a bell-shaped spectrum which is centered around 0 in the case of a spatial symmetric coupling, while in open-flow systems the maximum corresponds to the propagation velocity of perturbations [5]. Accordingly, the onset of chaos in a DDS strongly resembles the onset of a convective instability in a spatial system [9]. Moreover, the comoving exponent diverges logarithmically to $-\infty$ for $\alpha \to 0$ as seen from the inset, while it approaches a finite value for $\alpha \to \pi/2$. The latter value, which corresponds to a pure "spatial" propagation, can be simply obtained by neglecting the delayed contribution in Eq. (6). In

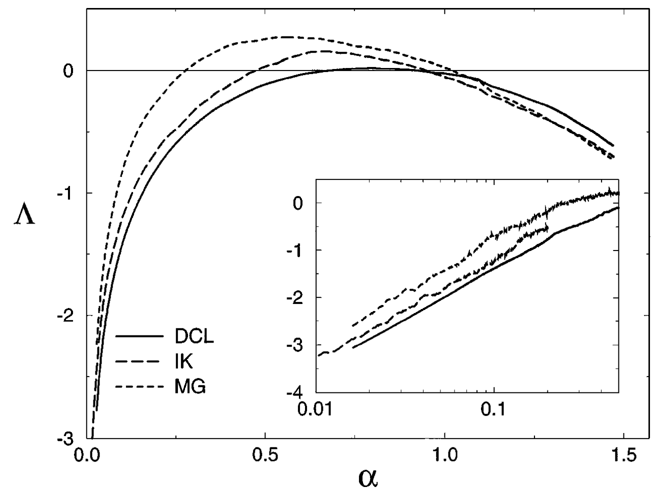


FIG. 1. Comoving Lyapunov spectra for the delayed complex Landau model with $\mu = -0.8$, $\eta = 1$, $\beta = 3$, Ikeda model with $a = 3$ and $z_0 = 0$, and Mackey-Glass equation for $b = 3$. The delay time is always $T = 100$. Logarithmic tails are reported in the inset.

principle, $\Lambda(\pi/2)$ can be positive and, in such a case, it coincides with the anomalous exponent as defined in Ref. [10]. This exponent has a special meaning in that it measures the tendency for the signal over a given delay unit to synchronize with the evolution in the previous unit.

The simplest framework where the deep analogy between DDS and 1D SES can be revealed is offered by the dynamics in the vicinity of a bifurcation, which typically factorizes in the product of a slow and a fast evolution. As a consequence, one can get rid of the discreteness of the "time" variable θ. We recall that this is precisely the experimental situation of Ref. [3]. The above qualitative considerations can be put on a quantitative basis by extending the method of amplitude equations [11] to a DDS. This allows us to show that the dynamics of a DDS in the vicinity of a Hopf bifurcation corresponds to that of a complex Ginzburg-Landau (CGL) equation in a suitable moving frame the velocity of which is determined self-consistently. For the sake of simplicity, we limit ourselves to describe the reduction method with reference to model (3), introduced in Ref. [3] to account for the experimental findings.

As indicated by the comoving analysis, there is a preferred velocity for the propagation of information and disturbances, so that it is convenient to express the equation in the moving frame (ξ, τ) defined as $\xi \equiv \sigma - v\theta, \tau \equiv \theta$. In analogy with SES, we introduce the scaling ansatz corresponding to a Hopf bifurcation (without loss of generality, we assume that μ is real),

$$\mu = -\eta + \mu_1 \varepsilon^2, \tag{11}$$

where $\varepsilon \ll 1$ is the smallness parameter. The variable $z(\xi, \tau) \equiv y(\xi, \theta)$ can be formally expanded in powers of ε,

$$z(\xi, \tau) = e^{i\phi\tau} \sum_{j=1}^{\infty} \varepsilon^j Z^{(j)}(\varepsilon\xi, \varepsilon^2\tau) , \tag{12}$$

where $Z^{(j)}$ denotes the jth coefficient, ϕ is the phase of η, and the exponential term contains the fast time dependence. Finally, the delay term z_d can be expanded as

$$\begin{aligned} z_d(\xi,\tau) &= e^{i\phi(\tau-1)} \sum_{j=1}^{\infty} \varepsilon^j Z^{(j)}(\varepsilon\xi + \varepsilon v, \varepsilon^2\tau - \varepsilon^2) \\ &= e^{i\phi(\tau-1)} \sum_{j=1}^{\infty} \varepsilon^j [Z^{(j)}(\varepsilon\xi, \varepsilon^2\tau) \\ &\quad + Z_\xi^{(j)}(\varepsilon\xi, \varepsilon^2\tau)\varepsilon v \\ &\quad - Z_\tau^{(j)}(\varepsilon\xi, \varepsilon^2\tau)\varepsilon^2 + \cdots]. \end{aligned} \tag{13}$$

Inserting Eqs. (11)–(13) into Eq. (3), and setting the coefficients of the ε powers, we find that the first order term vanishes identically, while the next two orders yield

$$Z_\xi^{(1)} = |\eta| v Z_\xi^{(1)}, \tag{14}$$

$$\begin{aligned} Z_\xi^{(2)} &= \mu_1 Z^{(1)} - |\eta| Z_\tau^{(1)} + |\eta| v Z_\xi^{(2)} \\ &\quad + \frac{1}{2} |\eta| v^2 Z_{\xi\xi}^{(1)} - (1 + i\beta)|Z^{(1)}|^2 Z^{(1)}. \end{aligned} \tag{15}$$

Equation (14) is identically satisfied if we fix

$$v = \frac{1}{|\eta|}, \tag{16}$$

which is exactly the velocity of the maximum comoving exponent previously discussed. Substitution of Eq. (16) into Eq. (15) yields

$$\eta Z_\tau^{(1)} = \mu_1 Z^{(1)} + \frac{1}{2\eta} Z_{\xi\xi}^{(1)} - (1 + i\beta)|Z^{(1)}|^2 Z^{(1)}, \tag{17}$$

which is a Ginzburg-Landau equation with a real diffusion coefficient. This implies that we are in the Benjamin-Feir stable region [11] where spatiotemporal intermittency has been observed [12]. This provides a first indication of the good correspondence with the original model where the maximum comoving exponent has been found to be nearly zero.

It is interesting to notice that the phase factor ϕ does not play any role, as it measures the dephasing of the homogeneous solution over the whole lattice length T. In order to test the goodness of the above derivation we have simulated both Eqs. (3) and (17) for $\mu = -0.8$ and $\eta = 1$, reporting the patterns in Fig. 2. The pattern resulting from Eq. (3) has been presented in such a way as to remove the drift, i.e., by suitably finding the appropriate (ξ, τ) variables. The corresponding velocity is very close to the theoretical expectation $v = 1$. Moreover, the two patterns clearly reveal the same features with only a minor difference in the spatial scale of the various structures. However, in judging the quality of the agreement, one has to bear in mind that, besides the approximations in the derivation of Eq. (17), qualitatively different boundary conditions influence the two simulations: the rightmost value in a spatial configuration is connected to the leftmost one of the next configuration in the DDS [Fig. 2(a)], while periodic conditions are fixed in the SES [Fig. 2(b)].

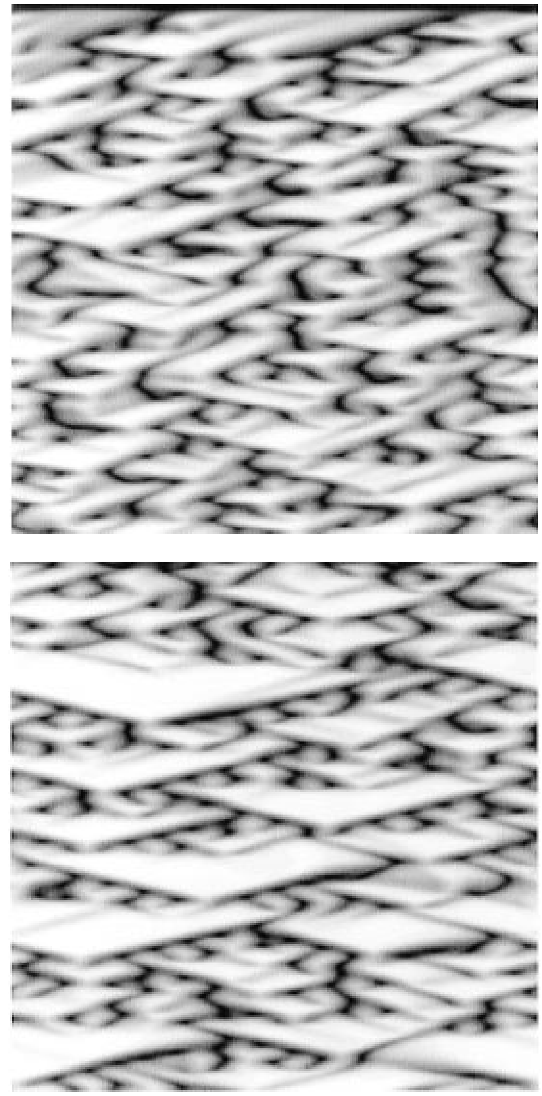
(a)
(b)


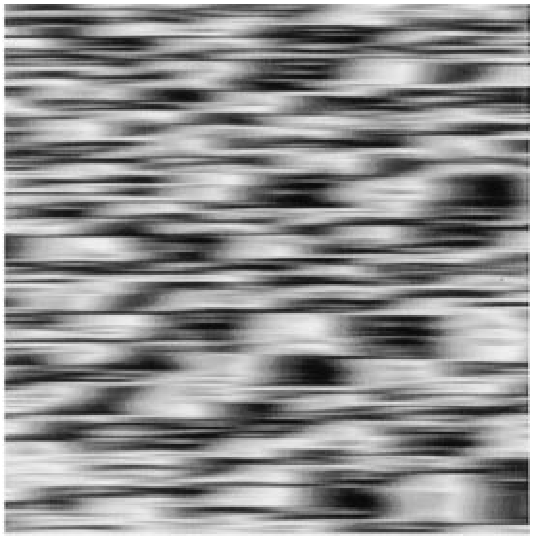
FIG. 2. Space-time representation for the delayed complex Landau model with the same parameter values as in Fig. 1. The modulus of y is plotted: (a) direct simulation in a frame moving with a velocity $v = \tan\alpha_0 = 1$ (see text); (b) integration of the corresponding CGL. Time ($1 \leq \theta \leq 500$) is increasing from top to bottom. The horizontal axis, representing the spacelike direction, corresponds to a delay unit T.

The above mentioned accord indicates that the mapping of a DDS onto a continuous-time model is effective not only in the vicinity of the Hopf bifurcation. However, there are systems where two consecutive spatial configurations are significantly different from one another, so that the discreteness of θ cannot be removed. This is the case of the pattern reported in Fig. 3(a), which refers to the Ikeda model (with $T = 100$ and $a = 3$), where one can see coherent structures coexisting with localized spatial regions characterized by clear temporal discontinuities.

The evolution from one to the next delay unit for a model of type (4) can be formally seen as the composition of two operators

$$y_d \rightarrow z_d = F(y_d) \rightarrow y = \mathcal{L} z_d, \tag{18}$$

(a)



(b)

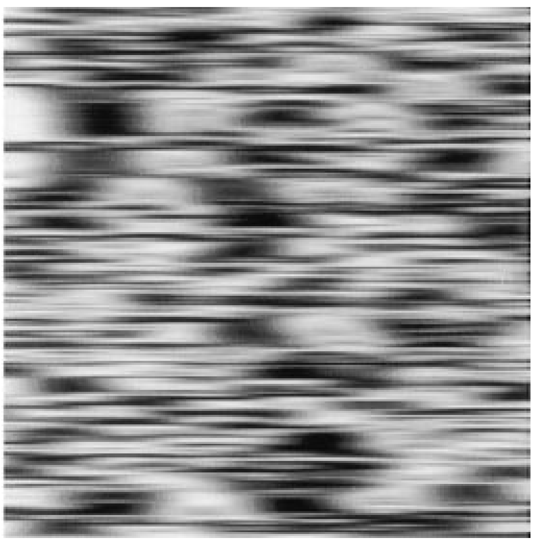


FIG. 3. Space-time representation for the Ikeda model with the same parameter values as in Fig. 1. Both patterns are plotted every fourth delay unit for 1300 delay units. For clarity reasons, only $1/3$ of a delay unit is reported. (a) The result of a direct simulation in a frame moving with velocity $v = \tan\alpha_0 = 1$ (see text); (b) results from the integration of the corresponding discrete time model.

where $\mathcal{L}$ is the propagator of the linear differential equation. The nonlinear operator F transforming y_d to the intermediate variable z_d acts exactly as the local map in standard lattices of coupled maps [13]. The action of $\mathcal{L}$ is better seen in Fourier space

$$y(\kappa) = \frac{z_d(\kappa)}{1 + i\kappa}, \tag{19}$$

where κ is the spatial wave number. Up to second order in κ, the above operator can be approximated by

$$y(\kappa) = z_d(\kappa)e^{-i\kappa}e^{-\kappa^2/2}, \tag{20}$$

where $e^{-i\kappa}$ is responsible for the shift from one to the successive delay unit, while the remaining Gaussian term corresponds to a diffusion in real space with a diffusion coefficient equal to 1. The result of the integration scheme, with $\mathcal{L}$ approximated by a normal diffusion process [i.e., by using Eq. (20) without shift], is reported in Fig. 3(b) showing a very good agreement with the original pattern in a moving frame with velocity $v = 1$. The same accuracy is found also for smaller and larger values of η where the evolution converges to a completely ordered and a more chaotic pattern, respectively. Therefore, we find again that the first two leading terms in the expansion of the spatial operator are responsible for the propagation of structures and for the diffusivelike properties.

Since both methods for approximating a DDS with a SES give rise, after removing the drift, to a spatially symmetric model, we can conclude that, at least in the regimes that we investigated, the strong asymmetry of the spectra in Fig. 1 with respect to their maximum value does not play a significant role in determining the properties of the evolution.

In this Letter we have shown on a quantitative basis that DDS can be assimilated to SES and that typical indicators of spatial systems, as the propagation of disturbances can be defined and provide interesting information about the instabilities occurring in this class of systems. Finally, because of the intrinsic discreteness of the time axis, DDS represent the closest physical models to lattices of coupled maps [13] and thus they represent serious candidates for the experimental observation of the many phenomena found in simulations on those abstract systems.

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