Spatiotemporal Stochastic Resonance in a ϕ^4 Model of Kink-Antikink Nucleation

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The synchronization and signal processing properties of a linearly coupled chain of N overdamped bistable elements, subject to a deterministic periodic signal and uncorrelated white noise, are addressed in the continuum limit of a ϕ^4 field theory. The scaling relations for the optimum noise and coupling strengths that correspond to the observed *spatiotemporal stochastic resonance* are derived via the ϕ^4 theory and shown to conform to the results of earlier numerical simulations in the large N limit.

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Recently, it has been shown that the stochastic resonance effect may be enhanced in a nonlinear dynamic element by coupling it (linearly) into an array of identical elements [1]; the output signal-to-noise ratio (SNR) may be maximized by treating the coupling and noise strengths as "design parameters," such that the condition of maximum output SNR corresponds to a spatiotemporal synchronization of the array dynamics to the external periodic signal. While this array enhanced stochastic resonance (AESR) is known to occur under somewhat more general conditions (e.g., global nonlinear coupling, correlated noise) [2], we consider here a linear chain of identical overdamped bistable elements with nearest-neighbor coupling:

$$\dot{x}_n = kx_n - k'x_n^3 + F\cos\Omega t + \epsilon(x_{n-1} - 2x_n + x_{n+1}) + \xi_n(t), \quad n = 1, 2, \ldots, N, \tag{1}$$

where the $\xi_n(t)$ are taken to be independent Gaussian delta-correlated noises having zero mean and variance $2D$, i.e., $\langle \xi_n(t)\xi_{n'}(0)\rangle = 2D\delta_{nn'}\delta(t)$. The signal amplitude F is always assumed to be subthreshold, i.e., the elements in the chain cannot switch between their stable steady states in the absence of the noise. Free boundary conditions are assumed throughout. The cooperation between noise and coupling has been shown [3] to lead to a *spatiotemporal stochastic resonance* characterized by the following behavior.

(i) The output SNR of a given element in the array shows a unique maximum on the plane (D, ϵ). The subscript m will be used here to denote the maximum SNR obtainable for a given value of N and the relevant values of the coupling and noise intensity. Furthermore, the SNR plotted *versus* D exhibits a typical SR behavior for any value of ϵ, namely, it peaks for a certain noise intensity $D(\epsilon)$, where $D(\epsilon)$ is an increasing function of ϵ.

(ii) The peak of SNR *versus* D for a coupled oscillator $(\epsilon > 0)$ is always more pronounced than for an uncoupled one $(\epsilon = 0)$. This result has been presented in [1] as an enhancement of the SR mechanism due to the coupling dynamics.

(iii) The spatiotemporal synchronization of the oscillators in the array is maximum at (D_m, ϵ_m), where the SNR reaches its maximum value SNR_m. The three quantities SNR_m, D_m, and ϵ_m depend crucially on the length N of the array. In [3] it has been observed that in the large N limit SNR_m approaches a constant, while D_m and ϵ_m scale like N and N^2, respectively (data points in Fig. 1).

In the present Letter we demonstrate that the results outlined in items (i)–(iii) can be easily interpreted as interrelated manifestations of SR in an overdamped ϕ^4 chain of finite size.

The $N \to \infty$ limit of Eq. (1) is commonly expressed in terms of the classical field $\phi(x,t)$ defined by the transformation $x_n(t) \to \phi(n\Delta x, t)$ with $\Delta x = 1/N$, whence the overdamped ϕ^4 theory [4,5]

$$\phi_t = k\phi - k'\phi^3 + \epsilon_N\phi_{xx} + F\cos\Omega t + \zeta(x,t), \tag{2}$$

where $\zeta(x,t)$ denotes a Gaussian noise source with spatiotemporal correlation $\langle\zeta(x,t)\zeta(x',t')\rangle = 2D_N\delta(t - t')\delta(x - x')$. The parameters ϵ and D of Eq. (1) are related to ϵ_N and D_N by $\epsilon = \epsilon_N N^2$ and $D = D_N N$, respectively. Under these assumptions, Eq. (1) describes the time evolution of an overdamped ϕ^4 chain of length 1 with homogeneous von Neumann boundary conditions, $\phi_x(0,t) = \phi_x(1,t) = 0$.

The ϕ^4 theory is known to bear both extended (phonons) and localized (soliton) solutions. Localized solutions can be well approximated to an appropriate

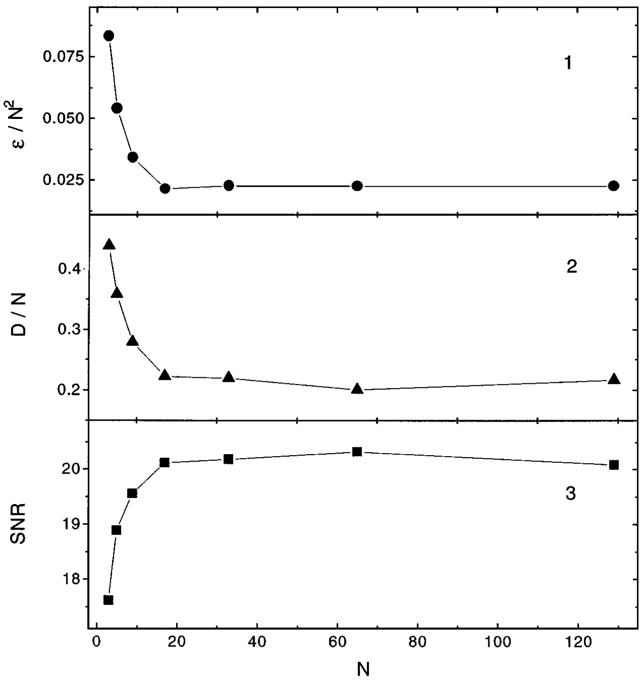

FIG. 1. Scaling behavior of optimal coupling (1), optimal noise (2), and best output SNR (3) obtained via numerical simulations of Eq. (1). $k = 210$, $k' = 1.47$, $F = 0.92$, $\Omega = 0.73$. The asymptotic ($N \to \infty$) values are in good agreement with predictions obtained via the ϕ^4 theory (see text).

linear superposition of moving kinks ϕ_+ and antikinks ϕ_-, with [4]

$$\phi_\pm = \tanh\left[\pm\frac{1}{2d}\,\frac{x - X_\pm(t)}{\sqrt{1 - \dot{X}_\pm^2(t)/c_0^2}}\right] \qquad (3)$$

provided that the separation between their centers of mass $X_\pm$ is very large compared with their size $d \equiv c_0/\omega_0$ (dilute gas approximation) and finite-length effects are negligible $d \ll 1$. In Eq. (3) we made use of dimensionless units. To make contact with Eq. (2) one should set $A = k/k'$, $\omega_0^2 = 2k$, $c_0^2 = \epsilon_N$ and rescale ϕ back to $\phi/\sqrt{A}$. Equation (2) with $F = 0$ provides a simple, efficient thermalization mechanism for the ϕ^4 theory in the overdamped limit [6]. The equilibrium kink (antikink) density in a ϕ^4 theory at finite temperature is [4]

$$n_\pm = n_0 = \left(\frac{3}{2\pi}\right)^{1/2}\frac{1}{d}\left(\frac{E_0}{D_N}\right)^{1/2}\exp(-E_0/D_N)\,, \quad (4)$$

where $E_0 = 2A\omega_0 c_0/3$ is the rest energy and $M_0 = E_0/c_0^2$ is the mass of $\phi_\pm$. It follows that the dilute gas approximation holds for $n_0^{-1} \gg d$, that is, for low noise intensity, $D_N \ll E_0$. In such a regime, $\langle\dot{X}_\pm^2\rangle = D_N/M_0 \ll c_0^2$, so that the relativistic boost factor in Eq. (3) may be safely approximated to unity.

In the absence of perturbations, $\phi_\pm$ is the kink (antikink) solution of the SG equation both in the underdamped (with $\dot{X}_\pm$ constant) and in the overdamped limit (with $X_\pm$ constant). The perturbation terms $\zeta(x,t)$ and $F(t) \equiv F\cos\Omega t$, introduced in (2), cause a rigid translation of the kink (antikink) against which $\phi_\pm$ is in neutral equilibrium. In other words, the shape of $\phi_\pm$ is not affected by the perturbation, whereas its center of mass $X_\pm$ becomes a random variable. Hence, in the overdamped limit (2), a single kink (antikink) undergoes a driven Brownian motion described by the Langevin equation [6]

$$\dot{X}_\pm = \mp 2(F_A/M_0)\cos\Omega t + \eta(t)\,, \qquad (5)$$

where $F_A \equiv \sqrt{A}\,F$ and $\eta(t)$ is a Gaussian, zero-mean valued random force with correlation function $\langle\eta(t)\eta(0)\rangle = 2(D_N/M_0)\delta(t)$. The periodic forcing term pulls $\phi_\pm$ in opposite directions.

The elementary mechanisms which allows a ϕ^4 chain to switch between its vacuum configurations $\phi = \pm 1$ is the *nucleation of kink-antikink pairs.* [Note that the $\phi_\pm$ solutions (3) carry opposite topological charge and, therefore, they may only be created by the pair.] Thermal fluctuations are expected to trigger the process by activating a critical nucleus [7–9], the size of which may be shown to increase with decreasing F [7]. Provided that the critical nucleus size is small enough to ignore many-body effects [10,11] due to the thermalized kinks and antikinks with density (4), we can describe the nucleation process as a local two-body process. This picture requires that $F_A d \gg D_N$ and $E_0 \gg D_N$. The two-body nucleation mechanism can then be treated as an extension of the Kramers theory of thermally activated processes to multidimensional systems with neutral equilibrium (or zero) modes [9]. When the nucleation mechanism outlined here is compared with the hopping mechanism of an uncoupled oscillator over the potential barrier, the role of the linear coupling between nearest neighbor oscillators becomes apparent [12]: *The saddle-point configuration of the ϕ^4 chain is represented by a critical nucleus and not by the unstable homogeneous solution $\phi = 0$.*

The nucleation rate, defined as the number of kink-antikink pairs nucleated per unit of time and length, can be calculated analytically to a good approximation in the strong-forcing and low-noise limit introduced above. In passing we note that these are the conditions simulated in the numerical investigations of Refs. [1] and [3]. For simplicity, we assume that the chain sits initially in the stable homogeneous state $\phi_0 = -1$ and that the forcing term $F(t)$ is constant and positive definite. The $F(t)$ time dependence will be accounted for at a later stage, only, according to the prescriptions of the standard rate theory [9]. A large nucleus $\phi_N(x,X)$ with length $2X \gg d$ is well represented by the linear superposition of a kink and an antikink centered at $\mp X$, respectively,

$$\phi_N(x,X) = \phi_+(x + X,0) + \phi_-(x - X,0) + 1\,. \quad (6)$$

The center of the nucleus has been set at the origin without loss of generality. The components of a large nucleus experience two contrasting forces: an attractive force due to the vicinity of the nucleating partner and a repulsive force due to the external bias F. In view of Eq. (6), the potential function corresponding to the internal force is [13]

$$V_N(X) = \int_{-\infty}^{+\infty} H[\phi_N(x,X)]\,dx = 6E_0[(-2/3 + 3K - 2K^2) + (X/d)(1 - 3K^2 - 2K^3)], \tag{7}$$

with $K = \tanh^{-1}(X/d)$ and for $X \gg d$ may be further approximated to [5]

$$V_N(X) = 2E_0[1 - 6\exp(-2X/d)]. \tag{8}$$

The effective external potential can be read out directly from the drift term of Eq. (3), that is, $\pm 2F_A X$. The *critical* nucleus configuration $\phi_N(x,R)$ is attained for a relative kink-antikink distance $2R(F)$ such that the two competing forces compensate each other, i.e., for

$$2R(F) = -d\ln(F_A d/12E_0) = -d\ln(F/24\sqrt{3}F_c), \tag{9}$$

with $F_c = (2/3)\sqrt{k^3/3k'}$. The critical nucleus admits only one unstable mode, associated with the collective variable $X(t)$, with negative eigenvalue [13]

$$\lambda_0^N = V''(R)/M_0 = -4F_A/dM_0 = -6F/\sqrt{A}. \tag{10}$$

Moreover, its energy $\Delta E_N(F)$ is obtainable through Eq. (7) or (8) after replacing X with $R(F)$.

The nucleation rate in a biased overdamped ϕ^4 chain compact form [9,13] reads

$$\Gamma = \frac{|\lambda_0^N|}{\pi}\frac{Z_N}{Z_0}\exp(-\Delta E_N/D_N), \tag{11}$$

where Z_0 and Z_N denote the partition function for the vacuum and the critical nucleus field configuration, respectively. The entropic factor Z_N/Z_0 accounts for both the phonon modes (with continuum spectrum), which "dress" ϕ_0 and ϕ_N, and the two internal modes of ϕ_N with discrete (nearly degenerate) eigenvalues $\lambda_b^N = 3\omega_0^2/4$ [4,5]. A standard calculation yields [14]

$$\frac{Z_N}{Z_0} = 9\frac{\omega_0^2}{\lambda_b^N}\left(\frac{\Delta E_N}{D_N}\right)^{1/2}\left(\frac{\omega_0^2}{2\pi|\lambda_0^N|}\right)^{1/2}. \tag{12}$$

On putting Eqs. (7)–(12) together we finally arrive at our analytical expressions for Γ:

$$\Gamma = \frac{8}{\pi}\frac{\omega_0^2}{d}\left(\frac{3F}{\sqrt{2}F_c}\right)^{\frac{1}{2}}\left(\frac{\Delta E_N}{D_N}\right)^{\frac{1}{2}}\exp(-\Delta E_N/D_N). \tag{13}$$

To account for the time dependence of the external bias $F(t)$, we assume with Ref. [3] that for $D_N \lesssim (D_N)_m$ the angular frequency Ω is much larger than Γ. This means that the effective rate Γ_N is given by the time average of Γ, (13), over one forcing period $2\pi/\Omega$. When taking such an average, one should notice that for the stable configuration $\phi_0 = -1$ (or $\phi_0 = 1$) and $F(t) \le 0$ [or $F(t) \ge 0$] Γ vanishes. Furthermore, the result of time averaging over the semiperiod with $F(t) > 0$ [or $F(t) < 0$] is crudely reproduced by replacing F with $F/\sqrt{2}$ both in the Arrhenius factor and in the prefactor of Eq. (13), so that $\Gamma_N(F) = (\frac{1}{2})\Gamma(F/\sqrt{2})$. The same treatment applies to the effective length $2R_N(F)$ of the critical nucleus, i.e., $2R_N(F) = R(F/\sqrt{2})$.

Going back to the finite-length, overdamped ϕ^4 chain simulated in [1,3], it is clear that in our picture the observed spatiotemporal synchronization of the chain would take place when the length of the critical nucleus is of the order of half the chain itself (as required by the free-end boundary conditions adopted there [15]). In the continuum units of Eq. (2), this amounts to the condition $2R_m = \frac{1}{2}$. On solving this equation for $\epsilon_N \equiv c_0^2$, one obtains from (9)

$$(\epsilon_N)_m = \frac{\epsilon_m}{N^2} = \frac{\omega_0^2}{16\ln^2(F/24\sqrt{6}F_c)}. \tag{14}$$

Under this condition, the SNR can be easily approximated by

$$\mathrm{SNR} = 10\log_{10}[\pi A\Gamma_N/16D_N^2], \tag{15}$$

where we have made use of the perturbation expression for the SNR of an uncoupled oscillator [16]. For a fixed value of the coupling ϵ_N, the SNR peaks for

$$\Delta E_N/D_N = 5/2 \tag{16}$$

[see Eq. (13)], thus showing a characteristic SR behavior. Note that the relevant activation energy is represented by the energy of the critical nucleus (7) and (8). Condition (16) implies that the SR value of D_N increases with $\sqrt{\epsilon_N}$, since to a rough approximation $\Delta E_N \sim 2E_0$ and E_0 is proportional to $\sqrt{\epsilon_N}$. This remark clarifies item (i).

The value of the noise intensity D_m, corresponding to the maximum synchronization of the chain, follows immediately by combining the SR condition (16) with the synchronization condition (14):

$$(D_N)_m = \frac{D_m}{N} = \frac{8}{15}\omega_0 A\left(\frac{\epsilon_m}{N^2}\right)^{1/2}\left(1 - \frac{F}{4\sqrt{6}F_c} + \cdots\right). \tag{17}$$

The results of Eqs. (14) and (17) reproduce the scaling laws observed numerically in [3]. On inserting the actual simulation parameter values $k = 2.10$, $k' = 1.47$, $\Omega = 0.73$, and $F = 0.92$, one obtains $\epsilon_m/N^2 = 0.021$ and $D_m/N = 0.18$ in reasonably good agreement with the numerical estimates (0.02 and 0.21, respectively) obtained via direct simulation of (1) (Fig. 1).

The corresponding value of SNR_m follows immediately by substituting the asymptotic values of ϵ_m/N^2 and D_m/N into Eq. (15). For the above parameter values, our prediction for SNR_m is of the order of 24, independent of the value of N, to compare with a simulation estimate of 21 (Fig. 1). At this point, we stress the fact that, since our approach to the overdamped ϕ^4 chain was developed in the continuum limit, our scaling laws for ϵ_m, D_m, and SNR_m are only tenable in the limit $N \rightarrow \infty$, as pointed out in item (iii) [17].

Finally, we remark that the statement in item (ii) comes about quite naturally under the present scheme of spatiotemporal synchronization. The maximum SNR for a coupled oscillator is larger than for an uncoupled one because the saddle-point configuration of the ϕ^4 chain involves an activation energy $\Delta E_N(F)$ smaller than the energy associated with the homogeneous configuration $\phi = 0$. The switching mechanism is thus enhanced for $\epsilon > 0$. On the other hand, for $\epsilon > \epsilon_m$ the critical nucleus would stretch beyond the boundaries, thus making the nucleation mechanism less efficient, as clearly shown in Fig. 1 of [1]. Ultimately, for very large values of ϵ the chain will behave like a rigid rod and the coupling becomes immaterial.

The array enhanced stochastic resonance (AESR) phenomenon and its attendant spatiotemporal synchronization may well be an important feature of information processing in neural networks. In fact, we believe the AESR phenomenon to be sufficiently general that it might be applicable to the design and operation of extended systems ranging from bioengineering receptors to remote sensing arrays.

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