Self-Affine Asperity Model for Earthquakes

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(Received 28 July 1995)

A model for fault dynamics consisting of two rough and rigid Brownian profiles that slide one over the other is introduced. An earthquake occurs when there is an intersection between the two profiles. The energy released is proportional to the overlap interval. Our model exhibits some specific features which follow from the fractal geometry of the fault: (1) nonuniversality of the exponent of the Gutenberg-Richter law for the magnitude distribution, (2) presence of local stress accumulation before a large seismic event, and (3) nontrivial space-time clustering of the epicenters.

PACS numbers: 91.30.Px, 05.40.+j

Many forms of scaling invariance appear in seismic phenomena: The celebrated Gutenberg-Richter law for the magnitude distribution [1], the Omori law for the time correlations of aftershocks [2], and space-time clustering of the epicenters [3] are common marks of the earthquake statistics. Unfortunately, the complexity of modeling the motion of a fault system, even in rather well-controlled situations, such as the San Andreas fault in California, is a highly difficult task, and it is still controversial what the correct theoretical framework is at the very origin of scaling laws. It is thus important to individuate models as simply as possible that are able to exhibit the main qualitative features of the fault dynamics. Their physical relevance stems from the specific predictions on the *real* seismic activity, which might be verified from experimental data.

One of the first attempts in this direction is due to Burridge and Knopoff [4], who introduced a stick-slip model of coupled oscillators to mimic the interaction of two fault surfaces. In practice, one considers blocks on a rough support connected to each other by springs. They are also connected by other springs to a driver which moves at a very low constant velocity. The blocks stick until the spring forces overwhelm the static friction and then one or more blocks slide, releasing an "earthquake" energy proportional to the sum of the displacements. A numerical integration of the Newton equations for a one-dimensional chain with a large number of homogeneous blocks has been shown to exhibit the Gutenberg-Richter law [5] (see also [6] for the connection with the chaotic behavior of the system). Moreover, it has been proposed that the qualitative aspects of earthquakes (and of Burridge-Knopoff models) are captured by the so-called sandpile models, which represent the paradigm of a large class of self-organized critical (SOC) systems [7], where the scaling is spontaneously generated by the dynamics. In fact, there is a whole generation of SOC models to explain the scale invariant properties of earthquakes [8,9]. These types of models suggest, however, that there is no stress accumulation before a big earthquake and the exponent of the Gutenberg-Richter law is expected (with some exceptions [10]) to be universal. In addition, the space-time distribution of the epicenters has no clear relation with the experiments where nontrivial clustering and correlations are present.

In order to go beyond these limitations we follow here an alternative approach where the critical behavior is not self-organized but stems from the fractal geometry of the fault that is supposed to arise as a consequence of geological processes on very long time scales with respect to the seismic dynamics. Looking at the system on the time scale of human records the fault structure can be considered assigned and just slightly modified by earthquakes.

Many authors pointed out that natural rock surfaces can be represented by fractional Brownian surfaces over a wide scale range [11,12] and that also the topographic traces of the fault surfaces exhibit scale invariance [13]. A fault can thus be regarded as a statistically self-affine profile $h(x)$, whose height scales as $|h(x + \ell) - h(x)| \sim \ell^H$. In $d = 2$, such a profile $h(x)$ can be generated by fractional Brownian motion with the exponent H and in $d = 3$ by the standard generalization given by Brownian reliefs [14,15]. The exponent $0 \le H \le 1$ controls the roughness of the fault where the standard random walk profile corresponds to $H = \frac{1}{2}$, and a differentiable curve corresponds to $H = 1$. The fractal dimension of the profile is well known to be $D_F = d - H$. In this context, Huang and Turcotte [11] introduced a static model where the average of all the seismic events contributing to the Gutenberg-Richter law is taken over many uncorrelated realizations of one single fractal profile. The purpose of this Letter is to introduce a dynamical model, called the self-affine asperity model (SAM), that describes the seismic activity considering two profiles sliding one over the other instead of only one as in [11]. Such a model has the advantage of exhibiting strong spatial and temporal correlations also between far away seismic events, and allows us to infer some specific and new predictions about the relation between the roughness of the fault H and the scaling exponent of the Gutenberg-Richter law as well as on the spatiotemporal distribution of epicenters. It can be regarded, in a certain sense, as the limit of infinite rigidity of the Burridge-Knopoff models and thus represents an alternative limit with respect to the SOC models.

Operatively, the SAM is defined by the following dynamical rules: (i) We consider two profiles, say, $h_1(x)$ and $h_2(x)$, on parallel supports of length L at infinite distance. The initial condition is obtained by putting them in contact in the point where the height difference is minimal, so that $h_1 - h_2 \geq 0, \forall x \in [0, L]$ [see Fig. 1(a)]. (ii) The successive evolution is obtained by drifting a profile in a parallel way with respect to the other one at a constant speed v, so that $h_1(x;t) = h_1(x - vt)$. (iii) At each time step t, one controls whether there are new contact points between the profiles, i.e., whether $h_1(x;t) - h_2(x) < 0$ for some x value. An intersection represents a single seismic event and starts with the collision of two *asperities* of the profiles. The energy released is assumed to be proportional to the breaking area of the asperities, i.e., the extension of the hypersurfaces, in general, of dimension $d - 1$, involved in the collision of the asperities during an earthquake. For more sophisticated schemes we refer the reader to [16]. In the case $d = 2$ the energy released is given by the sum of the lengths of the two segments indicated with A and B in Fig. 1(b). (iv) We do not allow the developing of new earthquakes in a region where a seismic event is already taking place, i.e., with reference to Fig. 1(b); we do not take into account the earthquakes which eventually take place in the region A and B of the two profiles, until A and B have a nonzero overlap.

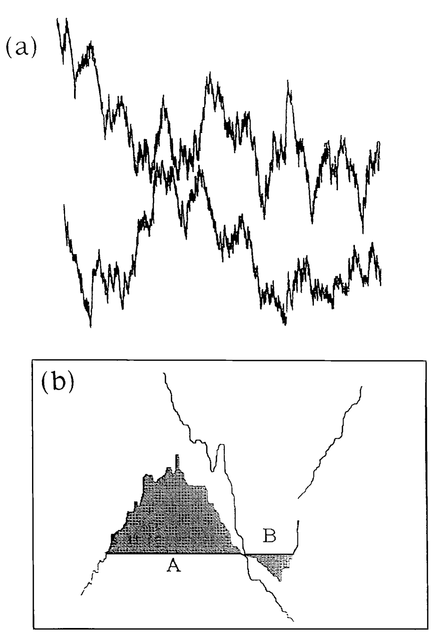

FIG. 1. (a) Example of two Brownian profiles modeling the fault surfaces. (b) Sketch for the definition of the energy released during an earthquake: It is assumed to be proportional to the breaking area [the $(d - 1)$-dimensional sets A and B] between the two asperities: $E \propto A + B$.

With these rules, the motion of the two profiles simulates the slipping of the two walls of a single fault. The points of collision are the points of the fault where the morphology prevents the free slip: These are the points where there is an accumulation of stress and, consequently, a rise in pressure. When the local pressure exceeds a certain threshold, it causes a breaking, an earthquake, which allows the stress to relax and the energy, previously accumulated, to redistribute all around.

For the sake of simplicity, in the SAM, there is no real breaking of the profiles as a consequence of an earthquake, and the profiles maintain their structures after a crash. It is possible to introduce a more realistic breaking mechanism where there is also a modification of the asperity form after an earthquake. However, we have verified that the main qualitative features remain unchanged. So we are in the opposite perspective than SOC models. In our case the earthquake dynamics has no effect on the structure of the profile. A realistic situation could well correspond to intermediate cases, of course.

It is worth stressing that the SAM exhibits a strong nonlocality, since a collision at a point x at the time t can trigger, at some later time, a subsequent event also very far away. One of the main advantages of the SAM consists in the possibility of deriving various analytic results using the properties of Brownian profiles. The most impressive characteristic of the earthquake statistics is the Gutenberg-Richter law. It states that the probability $P(E)\, dE$ that an earthquake releases an energy in the interval $[E, E + dE]$ scales according to a power law $P(E) \sim E^{-\beta-1}$ with an exponent β of the order of unity [10]. It is a controversial issue whether β is universal or varies in a narrow range according to the characteristics of the fault system.

In the framework of our model it is possible to relate the value of the exponent β to the geometrical properties of the faults. In particular, it can be shown that

$$\beta = 1 - H/(d - 1) = (D_F - 1)/(d - 1). \qquad (1)$$

This relation accounts for the direct dependence of the β exponent on the roughness of the faults H.

In order to derive (1), consider the profile $h_1(x;t) - h_2(x)$, which being given by the difference of two Brownian profiles is, in its turn, a Brownian profile at any time t. The statistics of the intersections between the two profiles is then given by the statistics of the intersections of the Brownian profile difference with a straight line along the temporal axis. Because of the invariance under temporal shifts of the profile, we can assume that the statistics of the intersections obtained at any time with a profile difference is given by the statistics of the intersections of an infinite profile with a zero level straight line.

In this perspective, a seismic event releases an energy proportional to the interval between two subsequent intersections between a Brownian profile and the zero level straight line. It is well known that the set obtained by the intersection between a fractional Brownian profile or relief of dimension $d - H$ embedded in a d-dimensional space and a hypersurface of dimensionality $d - 1$ is a fractal with a dimension given by the law of addition of the codimension [14], $(d - H) + (d - 1) - d$, so that the number of intersections in a hypersurface of volume $E \sim L^{d-1}$ scales as $N(L) \sim L^{d-1-H}$. Now if we identify the energy released from an earthquake with the size E of an intersection, we can determine the exponent β by consistency requirements. In our case the probability $P(E)$ is given by the probability of finding an intersection, of size E, between a d-dimensional surface and a $(d - 1)$-dimensional hyperplane.

It is quite natural to assume the existence of a power law $P(E) \sim E^{-1-\beta}$ of the Gutenberg-Richter type in our model as a consequence of its geometrical structure. It follows that the average value of the intersection size should also scale as

$$\langle E\rangle \equiv \int_0^{L^{d-1}} P(E)E\, dE \sim L^{(d-1)(1-\beta)}. \tag{2a}$$

Moreover, the typical length of a $(d - 1)$-dimensional interval is the total length L^{d-1} of the support divided by the number of intersections $N(L)$ so that we also obtain

$$\langle E\rangle = L^{d-1}/N(L) \sim L^{H}. \tag{2b}$$

The consistency of these two results thus implies $H = (d - 1)(1 - \beta)$ and so relation (1) between the Gutenberg-Richter exponent β and the roughness index H of the fault.

It is interesting to notice that the value $\beta = 1$ is an upper bound reached when the roughness of the fault is maximal ($H = 0$). Moreover, $\beta = 1$ is also recovered for all H values in the mean field limit $d \to \infty$, while, at $d = 3$, β can vary in the range [0.5, 1].

We have performed numerical simulations by considering two Brownian profiles, one of which is at rest and composed of 10^4 points and the other, slipping over the first one, composed of 2×10^4 points. In this way each realization of the dynamics lasts a time $T = 10^4$. The probability distribution of earthquakes has been obtained by averaging over many realizations of the dynamics. Figure 2 shows the numerical results in the case of $H = 0.5$ and $d = 2$. The exponent of the power law in this case is $\beta = 0.5$ in good agreement with our theoretical prediction. The Gutenberg-Richter law is obtained by the cumulative distribution of the frequency of earthquakes, i.e., the integral of the distribution shown in Fig. 2.

Another interesting feature that can be studied in the framework of the SAM is the phenomenology of the space-time correlations of earthquakes. In particular, we will focus on the problem of the spatial clustering of epicenters [17] and we refer the reader to [16] for a more exhaustive treatment of this point, including the analysis of the correlation functions and the temporal fractal distribution of epicenters. In our model the space location of an epicenter is defined in correspondence with the first point of contact of the two profiles. Numerical simulations performed on the SAM in the cases with $H = 0.3$, 0.5, and 0.7 would seem to provide a clear evidence of a spatial clustering of the epicenters on a set with a fractal dimension smaller than 1. For instance, a box-counting analysis on a system with linear dimension $L = 10^4$ with roughness exponent $H = 0.5$ gives a fractal dimension $d_{\mathrm{ep}} \simeq 0.78$. However, this result is a nontrivial finite size effect, since the set of epicenters tends to be compact when $L \to \infty$. In fact, it can be proved, for $H = 0.5$, that the fractal dimension $d_{\mathrm{ep}}(L)$ of the epicenter set in a fault of a linear size L is

$$d_{\mathrm{ep}}(L) \simeq 1 - \gamma \log\log L/\log L \quad \text{for large } L\,. \tag{3}$$

Let us, indeed, consider two Brownian profiles of length L as in Fig. 1(a). The distance $h_0(L)$ between the barycenter of the two profiles can be obtained from the iterated logarithm theorem [18], which states that, for a partial sum $S_n = \sum_{i=1}^{n} x_i$ of identically distributed random variables x_i with $\langle x_i\rangle = 0$ and $\langle x_i^2\rangle = 1 \forall i \in 1,\ldots,n$, the following holds:

$$P\left(\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right) = 1\,. \tag{4}$$

That means the maximum $M(L)$ of a Brownian profile scales as $M(L) \sim \sqrt{2L\log\log L}$.

Now the distance $h_0(L)$ is given exactly by the maximum value of a Brownian profile obtained by the difference of two Brownian profiles, that is, $h_0(L) \sim M(L)$. On the basis of this result, it is possible to estimate how the number of epicenters scales as a function of L. Considering the configuration where two Brownian profiles are $h_0(L)$ apart, the number of points of the lower profile at a certain height h with respect to its barycenter is

$$N_{\mathrm{down}} \sim \sqrt{L}\, \exp[-(h^2/2\eta L)]\,, \tag{5}$$

where η is a constant depending on the value of $\langle x_i^2\rangle$ [16]. We now have to integrate over all the possible values of h that correspond to the heights at which there could be an intersection of the two profiles in order to obtain the number of events (N_{ep}). The two integration extremes are given by the maximum value of the lower profile and the minimum value of the upper one, that is,

$$N_{\mathrm{ep}}(L) \sim \sqrt{L} \int_{h_0-\sqrt{2L\log\log L}}^{\sqrt{2L\log\log L}} \exp -\left(\frac{h^2}{2\eta L}\right) dh \sim \frac{L}{(\log L)^{\gamma}} \sqrt{\log\log L}\,, \tag{6}$$

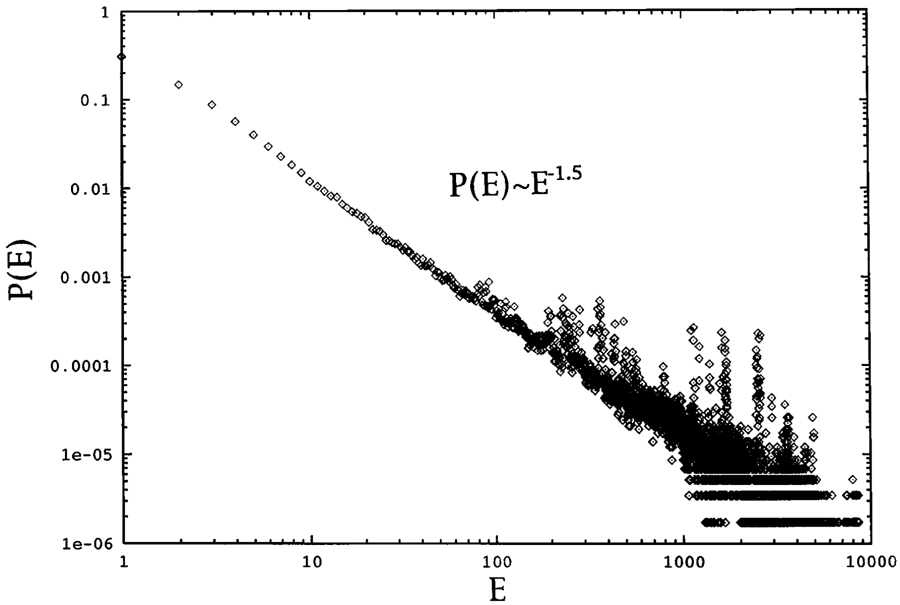

FIG. 2. Probability density of the earthquakes releasing an energy E vs E for roughness index $H = 0.5$.

where $\gamma = \alpha/\eta$ and α is an intermediate value between $\sqrt{2} - 1$ and 1. Using the mass-length definition of fractal dimension, $d_{\mathrm{ep}}(L) = \log N_{\mathrm{ep}}(L)/\log L$, relation (3) is proved. The asymptotic value $d_{\mathrm{ep}} = 1$ is reached very slowly at increasing L, and it cannot be detected but by huge simulations. We have checked the validity of (5) for profiles with a linear size L varying in the range 10^2–10^6. Work is in progress to extend our results to the case of a generic roughness index H [16].

In summary, we have proposed a model of earthquakes where the critical behavior is generated by a preexistent fractal geometry of the fault. The statistics of earthquakes is thus related to the roughness of the fault via the scaling relation (1) between critical indices. This result suggests that the younger the fault system, the larger the β exponent, since the roughness of a fault is expected to decrease in geological times. The exponent β, therefore, is nonuniversal. The model exhibits complex space-time correlations between epicenters: From the temporal point of view, there exists a fractal clusterization [16], although the spatial fractal distribution of the epicenters turns out to be a finite size effect that is very difficult to detect from data analysis. Our model provides a possible explanation for the highly irregular and nonrandom distribution of epicenters that is experimentally observed. Moreover, the accumulation of pressure is at the very origin of large seismic events in the SAM. The presence of such an effect could be tested also in real situations, e.g., by piezoelectric measurements.

We are grateful for interesting discussions to E. Caglioti, O. Mazzella, S. Solla, and R. Scarpa.

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