Complex Fractal Dimensions Describe the Hierarchical Structure of Diffusion-Limited-Aggregate Clusters

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We analyze large diffusion-limited aggregates and uncover a *discrete* scaling invariance in their inner structure, which can be quantified by the introduction of a set of *complex* fractal dimensions. We provide a theoretical framework and prediction of their values based on renormalization group theory and a previous wavelet analysis.

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In the context of the renormalization group theory of critical phenomena, self-similarity at the critical point is characterized by an invariance with respect to arbitrary magnifying factors, i.e., *continuous* scale invariance. In contradistinction, a system exhibits discrete scale invariance (DSI) if it is invariant under a *discrete* set of dilatations only. Formally, DSI leads to *complex* exponents and log-periodic corrections to scaling [1].

DSI has long been restricted to "man-made" systems, like regular lacunar fractals and hierarchical systems, where it is built in their mere definition. To our knowledge, the first theoretical suggestion of its relevance to physics was put forward by Novikov [2] to describe the small-scale intermittency in turbulent flows. More generally, some out-of-equilibrium systems presenting intermittency have recently been found to have complex exponents. Rupture in strongly disordered systems is a sort of dynamical critical point [3] exhibiting strong intermittency in the failure process: Logarithmic periodicities in the rate of acoustic emissions preceding the rupture have been measured for the case of pressure vessels composed of Kevlar and carbon-fiber-reinforced resin [4]. A similar behavior has been documented on foreshock activity preceding large earthquakes in California and the Aleutian Islands [5] and on fluctuations in the chloride and sulfate ion concentrations of groundwater issuing from wells located near the epicenter of the recent Kobe earthquake in Japan [6]. In addition, complex exponents appear also in the seemingly different contexts of ϵ-expansion calculations for disordered systems [7] and the statistics of "animals" (connected clusters on a percolation lattice) [8].

These examples prompt a reconsideration of the issue of DSI and complex exponents. In particular, one would like to understand when they occur and to have a tractable model at hand to study them. On the theoretical side, as discussed in [8], complex exponents are a natural property of nonunitary Euclidean field theories, relevant to describe most disordered systems (due to the replica averaging) as well as geometrical systems (due to the nonlocality). Such theories have a fixed point action, which is fully scale invariant, but some observables may have complex dimensions, implying that their Green functions are invariant under discrete rescalings only. In these systems, DSI is "spontaneously generated" by the dynamics of the system. For instance, in disordered systems, DSI is probably associated with the spontaneous breaking of replica symmetry, which entails a hierarchical organization of phase space [9]. The fact that disorder can lead to the spontaneous appearance of a hierarchy of scales can easily be seen in one-dimensional Brownian motion in random media [10].

We consider the diffusion-limited-aggregation (DLA) model [11] which allows for a much more complete numerical analysis than animals and is the archetype of complex fractal growth phenomena to which the foregoing rupture problems belong [12]. We show by a careful numerical analysis that the interior of DLA clusters does exhibit DSI spontaneously and is characterized by a discrete set of complex fractal dimensions, the first being the usual real $D_0 = 1.65 \pm 0.05$ [13–15]. The others control the log-periodic fluctuations in the variation of the mass contained in a disk as a function of its radius. The appearance of DSI is then connected with the results of a recent wavelet transform analysis [13]. These are used to compute the spectrum of the complex dimensions, in good agreement with numerical results.

The off-lattice DLA clusters that we study are constructed by making a single particle of size l diffuse from a large distance until it makes contact with the boundary of the cluster at which it sticks to form a new boundary and so forth. A characteristic feature of DLA is that most of the growth takes place in an active zone near the outer radius of the cluster. This active zone moves outward, leaving behind an extinct, in an asymptotic sense, region. This screening of the inner region by the tips is the basic reason for the fractal branching of DLA growth. To quantify such an edifice, we thus need large DLA clusters ($M = 10^6$ particles in the present study) so that the inner

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inactive region contains several (here about six) generations of branching.

We start by presenting numerical evidence for log-periodic corrections to the main scaling. For this we construct the local dimension defined by $D_r(\log r) = d\log M(r)/d\log r$. A typical realization is shown in Fig. 1, where it is clear that D_r is not a constant in the scaling regime $l < r < R$ (where R is the radius of gyration of the cluster), but oscillates around the asymptotic value $\simeq 1.65$. From the expression of physical quantities in systems exhibiting DSI, we make the ansatz

$$M(r) \propto r^{D_0}[1 + C\cos(2\pi f_2 \log r)]. \qquad (1)$$

The use of the index 2 for f_2 will become clear from Eq. (5) below. Assuming that the correction is weak, we obtain $D_r(\log r) \simeq D_0 - 2\pi C f_2 \sin(2\pi f_2 \log r)$. The ansatz (1) can obviously be generalized to include further (damped) oscillatory components. In what follows, we will clearly identify another frequency f_1. To investigate the validity of Eq. (1), a Savitzky-Golay smoothing filter was applied to the data $M(r)$ in order to obtain a good numerical estimate of the local derivative D_r. This filtering method approximates the data locally (corresponding to some user-chosen window) with an nth degree polynomial preserving up to the nth moment of the data. Hence it has the advantage over, for instance, a moving average filter that the magnitude of the variations in the data, i.e., the value of the local extrema, is preserved to a large extent. This means that the amplitude of the expected oscillations is not severely reduced and can be measured accurately. Figure 1 is an example obtained with a window of ≈ 0.5 and using a six-degree polynomial. (We have checked that our results are robust with respect to reasonable changes of these parameters.) To extract the frequency(ies) of the observed oscillations, a Lomb (normalized) periodogram was used. This method makes a local least-square fit of the data to a simple cosine function (with a phase) for a given frequency range. It was preferred to other standard techniques, such as the fast Fourier transform, due to its impressive ability of finding periodic structures in small noisy data sets of unevenly sampled data. We analyzed 350 DLA clusters of 10^6 particles and found systematically that their periodogram presents two main peaks. The frequencies f of these two main peaks were recorded and the histogram of these frequencies is shown in Fig. 2. Two peaks at $f_1 \simeq 0.6 \pm 0.1$ and $f_2 \simeq 1.3 \pm 0.1$ are clearly visible and their existence is statistically significant (the difference between the maxima and minimum is larger than two standard deviations).

The appearance of DSI in DLA can be explained from previous results on wavelet analysis, and the frequencies f_1 and f_2 estimated. To do so, we assume that the DLA inner structure can be characterized by a critical point occurring at $\frac{l}{r} \rightarrow 0$ (with $r \ll R$). We use the results of the wavelet transform modulus maxima (WTMM) representation, which has been applied to the DLA azimuthal Cantor set [13]. This set is defined by the intersection of a DLA's inner-cluster structure with a circle of radius $r_0 \ll R$ (in practice, $r_0 = 480$ particle sizes, containing $\approx 8 \times 10^4$ particles for $M = 10^6$ DLA clusters). In the limit of $r_0 \rightarrow +\infty$ ($r_0/R \ll 1$), this set becomes a genuine Cantor set of zero measure and fractal dimension $D_0 - 1$, defined by $M(r) \sim r^{D_0}$ for $l < r < R$. Using the WTMM representation, the hierarchical structure of DLA azimuthal Cantor sets can be modeled by a multiplicative process [13] expressed mathematically by the piecewise linear hyperbolic map:

$$T(x) = \begin{cases} \lambda x, & 0 < x < \lambda^{-1}, \\ \lambda^2(x - 1) + 1, & 1 - \lambda^{-2} < x < 1, \end{cases} \qquad (2)$$

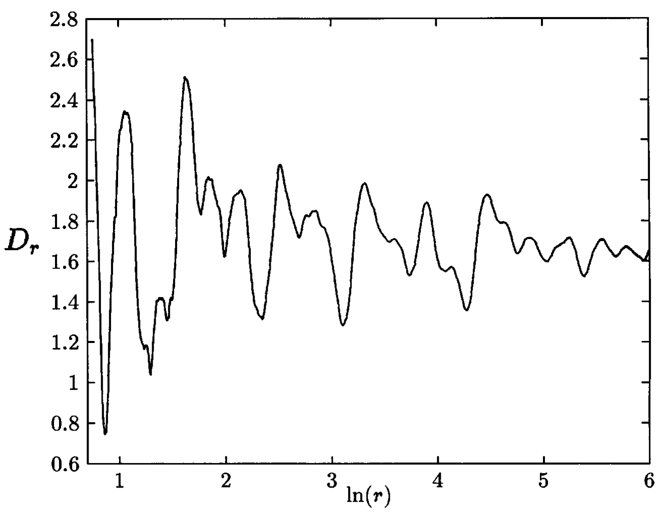

FIG. 1. Example of the local dimension $D_r(\log r) = d\log M(r)/d\log r$ as a function of $\log r$ for a typical DLA cluster. The numerical estimate of the derivative has been obtained with a Savitsky-Golay smoothing filter.

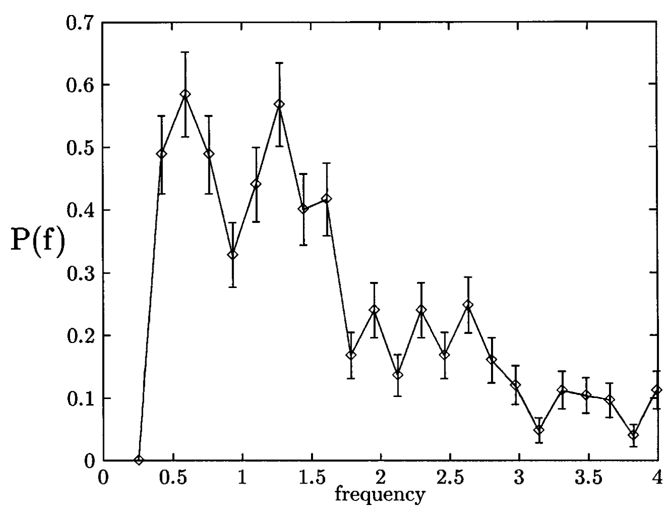

FIG. 2. Histogram of the frequencies found in the local dimension $D_r(\log r) = d\log M(r)/d\log r$ as a function of $\log r$, as illustrated in Fig. 1. The frequencies have been obtained by calculating Lomb's periodogram on D_r.

with $\lambda = 2.2 \pm 0.2$. The Cantor sets are then its invariant measure. This parametrization (2) of the WTMM representation is used in the following to obtain quantitative predictions.

Let $\mathcal{N}(a)$ be the number of maxima of the WTMM skeleton at scale a. This number roughly corresponds to the number of branches of the DLA cluster at scale a and at a distance r_0 from the center. From Eq. (2), it is straightforward to show that $\mathcal{N}(a)$ satisfies the "Fibonacci rule" [13]:

$$\mathcal{N}(a) = \mathcal{N}(\lambda a) + \mathcal{N}(\lambda^2 a). \tag{3}$$

Using our assumption of a critical point and $M(r) \sim r\mathcal{N}(\frac{1}{r})$, Eq. (3) translates into

$$M(r) = \lambda Mx\left(\frac{r}{\lambda}\right) + \lambda^2 M\left(\frac{r}{\lambda^2}\right), \tag{4}$$

where we substituted $a \to \frac{1}{r}$. Of course, these equations are only approximate descriptions of the noisy structures inherent in the stochastic DLA growth process. Looking for a solution $M(r) \sim r^D$, we find $M(r) = \sum_{n=-\infty}^{+\infty} c_n r^{D_n}$, where

$$D_n = 1 + (-1)^n \frac{\log\phi}{\log\lambda} + i\frac{n\pi}{\log\lambda}. \tag{5}$$

$\phi \simeq 1.618$ is the golden mean and n an integer. The leading term of the series in $M(r)$ is $M(r) \simeq r^{D_0}$, with $D_0 = 1 + \log\phi/\log\lambda \simeq 1.65$, which recovers the known value for the fractal dimension of DLA clusters. D_0 is the real part of all even-order dimensions D_{2n}. Notice that the real parts of even-order and odd-order dimensions are codimensions of each other: $\mathrm{Re}(D_{2n+1}) = 2 - \mathrm{Re}(D_{2n}) \simeq 0.35$.

What Eq. (5) teaches us is the existence of modulations to the leading power law behavior of $M(r)$ in the form of periodic (in $\log r$) corrections to scaling. Indeed, the real part of a term like $r^{D^R+iD^I}$ is simply $r^{D^R}\cos(D^I \log r)$, showing that a log-periodic correction to the scaling of $M(r)$ amounts to considering a complex fractal dimension and vice versa. The existence of imaginary parts in the dimensions (5) stems from the discrete scaling explicit in the writing of Eqs. (3) and (4). This can be put in an even clearer way by noting that, in the limit $r \to \infty$, Eq. (4) reduces to

$$M(r) \simeq \phi\lambda Mx\left(\frac{r}{\lambda}\right). \tag{6}$$

This is nothing but a discrete renormalization group equation on the observable $M(r)$ showing that the flow acting on the "control parameter" r is simply related to the hyperbolic map $T(x)$, and the dominant scaling structure of the DLA clusters is governed by the Lyapunov numbers λ and λ^2 of $T(x)$. The solution of the RG equation is $M(r) \simeq r^D$, with D obeying

$$\frac{\lambda^{D-1}}{\phi} = 1, \tag{7}$$

whose general solution recovers the set $\{D_{2n}\}$.

In sum, two fundamental frequencies are predicted, namely, $f_1 = 1/2\log\lambda$ for D_{2n+1} and $f_2 = 1/\log\lambda$ for D_{2n}. The first frequency reflects the fact that the convergence towards the fixed point at $r \to +\infty$, corresponding to the existence of the second frequency, is alternate with a subharmonic frequency. Using the independent numerical estimation of $\lambda \simeq 2.2$ [13] leads to $f_1 \simeq 0.6$ and $f_2 \simeq 1.3$ in excellent agreement with the previous numerical results. These values translate into $D_1^I = 2\pi/2\log\lambda \simeq 4$ and $D_2^I = 2\pi/\log\lambda \simeq 8$ for the imaginary part of the new scaling exponents.

This simple model would predict that the amplitude of the leading log-periodic correction with frequency f_2 is constant [Eq. (1)], and only the next correction to scaling is damped, adding in Eq. (1) a term of the form $r^{-\delta}\cos(2\pi f_1 \log r)$, where $\delta = D_0 - D_1^R \approx 1.3$. This does not agree with Fig. 1 where an exponential damping (in $\log r$) of all the oscillations is clearly visible as the critical point is approached. The reason is that, up to now, we had assumed that the scaling factors λ and ϕ were constant. In fact, they fluctuate from sample to sample, and when the scale is changed. It is not clear how to take these fluctuations into account explicitly. However, various simple models of fluctuations [8] suggest the existence of a *renormalization* of the dimensions by the disorder, leading to the expression

$$M(r) \propto r^{D_0}[1 + Cr^{-\alpha}\cos(2\pi f_2 \log r)], \tag{8}$$

Here $\alpha = D_0 - D_2^{\prime R} > 0$, where we have allowed for the renormalization of the even-order dimensions ($D_2^{\prime R} < D_2^R$).

In order to study this effect, we have performed an average of all 350 DLA clusters of the absolute value

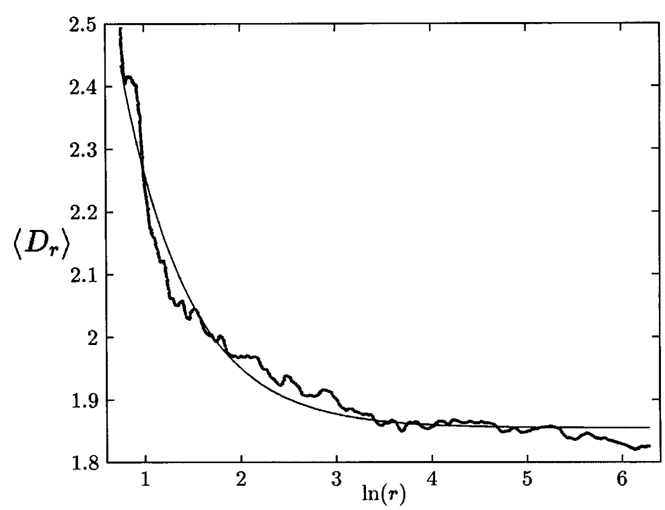

FIG. 3. The local dimension averaged as $\bar{D}_r = \langle |D_r(\log r) - \langle D_r(\log r)\rangle| \rangle + \langle D_r(\log r)\rangle$ as a function of $\log r$. The average is over more than 350 DLA clusters. The data have been fitted to the equation $a + be^{-\alpha \log r}$, giving $a \approx 1.80$, $b \approx 0.6$, and $\alpha \approx 0.7$.

of the oscillations. The quantity $\bar{D}_r(\log r) = \langle D(\log r)\rangle + \langle|D(\log r) - \langle D(\log r)\rangle|\rangle$ is shown in Fig. 3. As expected for finite samples the oscillations have been mostly erased, but we clearly observe the predicted exponential decay (in the $\log r$ variable). The best fit done in the range $\log r > 2$ and shown in the figure gives $\alpha \approx 0.7$, yielding $D_2^{IR} \approx 1$. For smaller $\log r$, we find a faster decay with a larger characteristic exponent $\alpha \simeq 1.3$, which corresponds probably to the correction $\delta = D_0 - D_1^R$ induced by the odd-order dimensions.

The renormalization of D_2^R into D_2^{IR} can be qualitatively understood as follows. The distribution of λ and ϕ [13] can be approximated by a product of three Gaussian functions, respectively, in terms of the variables $\log\lambda$, $\log\phi$, and $\log\frac{\phi}{\lambda^{D-1}}$ peaked, respectively, at $\lambda_0 = 2.2$, $\phi_0 = 1.6$, and $\frac{\phi_0}{\lambda_0^{D-1}} = 1$, with a common standard deviation $\sigma^2 = 0.23 \pm 0.03$. If we then make a random multiplicative renormalization group ansatz and consider the quantity $\langle M(r)\rangle$ averaged over this distribution, one can show [8] that Eq. (7) is replaced by

$$\left\langle\frac{\phi}{\lambda^{D-1}}\right\rangle = 1\,. \tag{9}$$

This yields the quantization equation

$$\frac{1}{2}\,\frac{\log(\phi_0/\lambda_0^{D_{2n}-1}) + (\sigma^2/2)[(D_{2n}-1)^2+1]}{1+(1/2)\,(D_{2n}-1)^2} = 2i\pi n\,,$$

with n integer. For $\sigma \rightarrow 0$, we recover the previous result for D_0. With the numerical values for σ, λ_0, and ϕ_0, Eq. (9) yields $D_0 = 1.9 \pm 0.3$ (obtained for $n = 0$), which, considering the crudeness of this model, is reasonable. For $n = 1$, we get $D_2 = 0.9 \pm 0.3 - i(7 \pm 1)$. The imaginary part is within the error bars equal to our previous measurement for f_2. The real part of D_2 is miraculously close to what we found numerically ($D_2^{IR} \simeq 1$) considering the crudeness of the ansatz.

Observe that, because of the disorder-induced renormalization of D_2^R, all corrections are damped and, in the limit of infinite sizes, continuous scale invariance is recoved in DLA.

The existence of the complex correction to scaling with a real part around 1 could be at the origin of the uncertainty remaining in the literature on the precise value of the DLA fractal dimension [14]. Indeed, depending on the techniques used to estimate the slope of log-log plots, one expects to get values for the fractal dimension close to but below the lead scaling.

The physical origin of the discrete scaling invariance of DLA clusters can be tracked back to the way successive screening of competing growing branches occurs. Take two neighboring branches of similar size. After a while, one will win over the other, screen it, and grow at its expense. As a DLA cluster grows, there is a succession of such rather brutal "discrete" screening events between the largest existing branches, and the size of the branches at which these intermittent screening events occur follow an approximate geometrical series of ratio λ. This picture, which should also apply for all branches at smaller scale, was suggested in [13] as a reasonable scenario leading to a Fibonacci structural ordering. Our analysis has shown that this geometric intermittent branching process leads to a discrete scale invariance.

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[15] The large value of the error bar accounts for the different estimates reported in the literature by many authors using various definitions of the fractal dimension; this uncertainty might also be the signature of a weak multifractal departure from homogeneous scaling invariance.